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SOME REMARKS ON BURZYŃSKI'S FAILURE CRITERION FOR ANISOTROPIC MATERIALS

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Some misstatements appearing in the final form of the failure criterion formulation, derived from Burzyński's hypothesis of material effort for anisotropic bodies, which haven't been noticed in the literature as yet, are pointed out and discussed. Alternative interpretations of the results obtained by Burzyński are presented. Propositions of different formulation of the failure criterion, basing on original ideas of Burzyński, are given.

Key words: Burzyński's hypothesis, limit condition, material effort, anisotropy.

1. Introduction

Among many propositions of the hypotheses of material effort for isotropic bodies, the one proposed by Burzyński in his doctoral dissertation (1928 [1]), surprises by its clear energy-based interpretation, variety of classes of materials it can be applied to and simplicity in formulation of the failure criterion, which can be determined only in terms of limit stresses under simple loads: uniaxial tension, compression and pure shear. Accounting for pressure sensitivity, Burzyński developed former ideas of his teacher, M.T. Huber [2], and anticipated later propositions of Drucker and Prager [3]. From the late twenties of the 20th century until now it remains one of the most general and practical propositions stated. However, it seems to be still underestimated, almost forgotten, especially abroad Poland. Extension of the given hypothesis accounting for anisotropy is even less known despite the fact that it was something completely new at that time – it could be compared only with some ideas introduced in the same year by Mises [4]. Both papers were published a few decades before other similar propositions by Hill (1948 [5]) or Hoffman (1967 [6]). Small popularity of the anisotropic version of Burzyński's condition is the reason for which it was not discussed as yet. In the current paper, some misstatements in the formulation given by Burzyński, which were not noticed and discussed in the literature, are pointed out. It is also the aim of the author to suggest alternative interpretation of the results obtained by Burzyński and to propose a formulation of the final limit condition, derived from Burzyński's hypothesis of material effort different from the original one.

2. Burzyński's hypothesis of material effort for anisotropic bodies

Burzyński considered an energy-based failure criterion, in which elastic energy density is expressed assuming linear dependence between the stress and strain states:

$$\begin{aligned} \mathbf{C}\boldsymbol{\sigma} = \boldsymbol{\varepsilon} \quad &\Rightarrow \quad C_{ijkl}\sigma_{kl} = \varepsilon_{ij}, \\ \mathbf{S}\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \quad &\Rightarrow \quad S_{ijkl}\varepsilon_{kl} = \sigma_{ij}, \end{aligned} \tag{2.1}$$

where $\mathbf{C}$ and $\mathbf{S}$ are fourth order symmetric compliance and stiffness tensor respectively, $\boldsymbol{\sigma}$ is the stress tensor and $\boldsymbol{\varepsilon}$ is an infinitesimal strain tensor. Assumption that Hooke's law is still valid even just before reaching the limit state, indicates that the limit state considered by Burzyński is in fact the limit of Hooke's law validity range – linear elasticity. All the limit stress quantities appearing in this formulation should be considered as the proportionality limit.

2.1. Hypothesis statement

Burzyński proposed to consider as a measure of material effort, the combination of distortional strain energy density and a part of volume change energy density, determined by function η, namely:

$$\Phi_f + \eta \cdot \Phi_v = K, \tag{2.2}$$

where K – limit value of energy density,

$$\Phi_v = \frac{1}{2}\mathbf{A}_\sigma \cdot \mathbf{A}_\varepsilon = \frac{1}{2}\left(\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{1}\right) \cdot \left(\frac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{1}\right),$$

$$\Phi_f = \frac{1}{2}\mathbf{D}_\sigma \cdot \mathbf{D}_\varepsilon = \frac{1}{2}\left(\boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{1}\right) \cdot \left(\boldsymbol{\varepsilon} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{1}\right),$$

$$\eta = \eta(p, \delta, \omega) = \left(\omega + \frac{\delta}{3p}\right).$$

$\mathbf{A}_\sigma$, $\mathbf{A}_\varepsilon$ and $\mathbf{D}_\sigma$, $\mathbf{D}_\varepsilon$ are spherical parts and deviators of stress and strain tensors respectively, $\mathbf{1}$ is an isotropic second rank symmetric tensor (identity tensor),

p denotes hydrostatic stress, and δ and ω are constant material parameters. The form of the function η was assumed by Burzyński.

Decomposition of the strain energy density into distortional and volumetric strain energy density is possible in general only for isotropic bodies or those of cubic symmetry. However, Burzyński stated that: *'Practically there are no physical reasons for which strain energy could not be decomposed into sum of two other energies, namely volumetric strain energy and distortional strain energy'* [1]. He considered a special class of materials of arbitrary symmetry, which the considered decomposition is always possible or equivalently – speaking in terms of tensor algebra – for which hydrostatic stress and dilatation are eigenstates of compliance and stiffness tensor respectively [7]:

$$\mathbf{C}\mathbf{1} = \Theta\mathbf{1} \quad \Rightarrow \quad C_{ijkl}\delta_{kl} = \Theta\delta_{ij} \quad \Rightarrow \quad C_{ijkk} = \Theta\delta_{ij}, \tag{2.3}$$

where Θ is the proportionality coefficient (eigenvalue of $\mathbf{C}$). This assumption leads to the following constraints on the components of compliance/stiffness tensor:

$$\begin{aligned}
&\text{(3 independent relations)} \quad \begin{cases} C_{1123} + C_{2223} + C_{3323} = 0, \\ C_{1131} + C_{2231} + C_{3331} = 0, \\ C_{1112} + C_{2212} + C_{3312} = 0, \end{cases} \\
&\text{(2 independent relations)} \quad \begin{cases} C_{1111} - C_{2222} = C_{2233} - C_{1133}, \\ C_{2222} - C_{3333} = C_{3311} - C_{2211}, \\ C_{3333} - C_{1111} = C_{1122} - C_{3322}. \end{cases}
\end{aligned} \tag{2.4}$$

These equations are called the *Burzyński's conditions*. If components of compliance or stiffness tensor of a given material satisfy the Burzyński's conditions (2.4), it is called the *volumetrically isotropic* material or simply the *Burzyński's material*. Total number of independent components of stiffness or compliance tensor, in case of volumetric isotropy, is reduced from 21 to 16.

2.2. Limit conditions

In case of isotropy, after substituting:

$$\begin{aligned}
\frac{1-2\mu}{1+\mu}\omega &= \frac{1-2\nu}{1+\nu}, \qquad & \frac{1-2\mu}{1+\mu}\delta &= \frac{3(k_c-k_r)}{1+\nu}, \\
12GK &= \frac{3k_ck_r}{1+\nu}, \qquad & \nu &= \frac{k_ck_r}{2k_s^2} - 1,
\end{aligned} \tag{2.5}$$

where G – Kirchhoff's modulus, μ – Poisson's ratio, k_c, k_r, k_s – limit values of stress at compression, tension and shearing tests, the general formulation of the

Burzyński hypothesis (2.2) can be expressed in terms of limit quantities, which are relatively easy to be measured:

$$\begin{aligned}(2.6)\quad &\sigma_{11}^2+\sigma_{22}^2+\sigma_{33}^2+2\left(1-\frac{k_ck_r}{2k_s^2}\right)(\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}+\sigma_{11}\sigma_{22})\\ &+\left(\frac{k_ck_r}{k_s^2}\right)(\sigma_{23}^2+\sigma_{31}^2+\sigma_{12}^2)+(k_c-k_r)(\sigma_{11}+\sigma_{22}+\sigma_{33})-k_ck_r=0.\end{aligned}$$

As it was said before, Burzyński also made an attempt to account for anisotropy in his hypothesis knowing that in fact, there are no ideally isotropic materials. He has considered a fully anisotropic material (except of its volumetric isotropy), yet for simplification of the criterion formulation he reduced the number of independent parameters. He used the so-called '*basic*' (or '*fundamental*') coordinate system, in which in expression of elastic energy density, the mixed terms involving shearing and normal stresses (or equivalently, linear and distortional strains) vanish. After certain rotation – which is 'only mathematically possible' [1] for volumetrically isotropic bodies – of a given coordinate system to the position, in which it can be considered as the 'basic' one, even in case of very low symmetries (total anisotropy, monoclinic symmetry, trigonal symmetry), the expression of the elastic energy density has the mathematical form at least as simple as in case of orthotropy. Such situation occurs when the Burzyński's conditions (2.4) are fulfilled and additionally, the following relations are true:

$$(2.7)\qquad \begin{aligned}C_{2223}\sigma_{23}-C_{1131}\sigma_{31}&=0,\\ C_{3331}\sigma_{31}-C_{2212}\sigma_{12}&=0,\\ C_{1112}\sigma_{12}-C_{3323}\sigma_{23}&=0.\end{aligned}$$

One can note that those conditions are fulfilled in case of a coordinate system with axes which are parallel to the directions of principal stresses, in which off-diagonal components of the stress tensor are always equal to 0. This is a very specific case – in fact there exist other basic coordinate systems, independent of the stress state. For example, in case of any material which is at least orthotropic (orthotropic, tetragonal, cylindrical, cubic), a coordinate system built on the axes of symmetry of such a material satisfies those conditions. In the basic coordinate system, elastic energy density can be expressed as follows:

$$\begin{aligned}(2.8)\quad \varPhi=&\underbrace{\frac{1}{2}B(\sigma_{11}+\sigma_{22}+\sigma_{33})^2}_{\varPhi_v}\\ &+\underbrace{\frac{1}{3}\big[L(\sigma_{22}-\sigma_{33})^2+M(\sigma_{33}-\sigma_{11})^2+N(\sigma_{11}-\sigma_{22})^2\big]+2P\sigma_{23}^2+2Q\sigma_{31}^2+2R\sigma_{12}^2}_{\varPhi_f},\end{aligned}$$

where

$$
\begin{aligned}
B &= \frac{1}{3}(C_{kk11} + C_{kk22} + C_{kk33}) \\
&\qquad = \frac{1}{3}(C_{11kk} + C_{22kk} + C_{33kk}), \quad k = 1, 2, 3 \quad \text{(no summation)}, \\
L &= \frac{3}{2}(B - C_{2233}), \\
M &= \frac{3}{2}(B - C_{3311}), \\
N &= \frac{3}{2}(B - C_{1122}), \\
P &= \frac{1}{4}\left(C_{2323} + 2C_{2331}\frac{C_{2223}}{C_{1131}}\right), \\
Q &= \frac{1}{4}\left(C_{3131} + 2C_{3112}\frac{C_{3331}}{C_{2212}}\right), \\
R &= \frac{1}{4}\left(C_{1212} + 2C_{1223}\frac{C_{1112}}{C_{3323}}\right),
\end{aligned}
\tag{2.9}
$$

B – bulk modulus, L, M, N, P, Q, R – generalized moduli of distortion.

Using the above formula in criterion (2.2) would give us the limit condition depending on 8 parameters, what makes it rather complex in analysis. In order to simplify it, Burzyński considered the strain energy density expressed only in terms of principal stresses – yet, he based on the assumption that $\sigma_1 \geq \sigma_2 \geq \sigma_3$ (or a set of inverted inequalities), what in simple load cases (uniaxial tests, pure shears) always guarantees that $\sigma_2 = 0$. In such a case, after further substitutions:

$$
\begin{aligned}
\frac{1 - 2\widetilde{\nu}}{1 + \widetilde{\nu}} &= \frac{3BM}{2LN}\omega, & \frac{3(k_c - k_r)}{1 + \widetilde{\nu}} &= \frac{3BM}{2LN}\delta, \\
\frac{3k_c k_r}{1 + \widetilde{\nu}} &= \frac{3KM}{LN}, & \lambda &= \frac{M^2}{2LN}, \\
\frac{M}{L} = \frac{M}{N} &= 2(1 - \lambda), & \varphi &= \sqrt{\frac{2(1 + \lambda)}{3}}, \\
\widetilde{\nu} &= \frac{1}{\varphi^2}\frac{k_c k_r}{2k_s^2} - 1, & \widetilde{\delta} &= \frac{1 + \widetilde{\nu}}{3}(1 - 2\lambda),
\end{aligned}
\tag{2.10}
$$

hypothesis (2.2) for anisotropic bodies can be written as a 4-parameter limit criterion, i.e. as:

$$(2.11)\quad \sigma_1^2+\left(\frac{(1-2\lambda)k_ck_r}{2(\lambda+1)k_s^2}+1\right)\sigma_2^2+\sigma_3^2+(k_c-k_r)(\sigma_1+\sigma_2+\sigma_3)$$
$$+2\left(1-\frac{k_ck_r(2-\lambda)}{2(\lambda+1)k_s^2}\right)\left[\sigma_2\sigma_3+\frac{(k_ck_r-2k_s^2)(\lambda+1)}{k_ck_r(2-\lambda)-2(\lambda+1)k_s^2}\sigma_3\sigma_1+\sigma_1\sigma_2\right]-k_ck_r=0.$$

Please note that while the limit condition proposed by Burzyński for isotropic bodies [1] is a scalar function of the first invariant of stress tensor and the second invariant of its deviator, function (2.11) can be no longer expressed in terms of only those two quantities. Proposition (2.11) can be considered as an extension of the limit condition for isotropic bodies, so that it accounted for the influence of the third stress tensor invariant; in this case, Lode angle dependence would be a result of distinct influence of the intermediate stress on the material effort. The influence of the Lode angle which is proportional to the third invariant of the stress deviator, can be easily observed on the plots of limit surfaces (in the space of principal stresses) which are no longer axi-symmetric surfaces.

2.3. Matrix form of the Burzyński limit condition for anisotropic bodies

The above limit condition (2.11) can be rewritten in such a matrix form:

$$(2.12)\quad \begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix}^T \begin{bmatrix}1 & \beta & \gamma\\ & \alpha & \beta\\ \text{sym} & & 1\end{bmatrix} \begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix} + \begin{bmatrix}(k_c-k_r)\\ (k_c-k_r)\\ (k_c-k_r)\end{bmatrix}^T \begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix} - k_ck_r=0,$$

where

$$\alpha=1+\frac{(1-2\lambda)k_ck_r}{2(\lambda+1)k_s^2},\qquad \beta=1-\frac{(2-\lambda)k_ck_r}{2(\lambda+1)k_s^2},\qquad \gamma=1-\frac{k_ck_r}{2k_s^2}.$$

Spectral decomposition of a linear matrix operator (which could be considered as a kind of a limit state tensor in the space of principal stresses), gives us an interesting result:

- One-dimensional subspace of hydrostatic stresses:
  Eigenvalue: $\chi_1=3-\dfrac{3k_ck_r}{2k_s^2(\lambda+1)}$, eigenstate: $\mathbf{h}_1=\dfrac{1}{\sqrt{3}}[1;1;1]$, $|\sigma_1|=p$.
- One-dimensional subspace of pure shears:
  Eigenvalue: $\chi_2=\dfrac{k_ck_r}{2k_s^2}$, eigenstate: $\mathbf{h}_2=\dfrac{1}{\sqrt{2}}[1;0;-1]$, $|\sigma_2|=\tau_{\max}$.

- One-dimensional subspace of deviators:
  Eigenvalue: $\chi_3 = \dfrac{3k_c k_r(1-\lambda)}{2k_s^2(1+\lambda)}$, eigenstate: $\mathbf{h}_3 = \dfrac{1}{\sqrt{6}}[1;-2;1]$, $|\sigma_3| = \tau_{45}$.

Contribution of certain stress states can be analyzed now. The first eigenstate $\mathbf{h}_1$ corresponds to the hydrostatic stress. Since inequality $\sigma_1 \geq \sigma_2 \geq \sigma_3$ is assumed, one can see that the second eigenstate $\mathbf{h}_2$ corresponds to maximum shear stress – please note that the contribution of this stress state to the total measure of material effort is independent of the anisotropy coefficient λ. The third eigenstate $\mathbf{h}_3$ is a composition of two non-orthogonal pure shears (we are considering classical scalar product defined as $\mathbf{A}\cdot\mathbf{B} = A_{ij}B_{ij}$); however, it is not a pure shear itself. Eigenstate $\mathbf{h}_3$ is orthogonal to the maximum shear state $\mathbf{h}_2$, but none of its pure shear components is orthogonal to $\mathbf{h}_2$. Please note that the inequalities $\sigma_1 \geq \sigma_2 \geq \sigma_3$ refer to the stress state $\boldsymbol{\sigma}$ itself, not to the projections of $\boldsymbol{\sigma}$ on chosen states, so it does not matter that those inequalities are not fulfilled in case of $\mathbf{h}_3$. Decomposition of the general stress state in the basis of eigenstates of the limit state operator, can be illustrated as shown in Fig. 1.

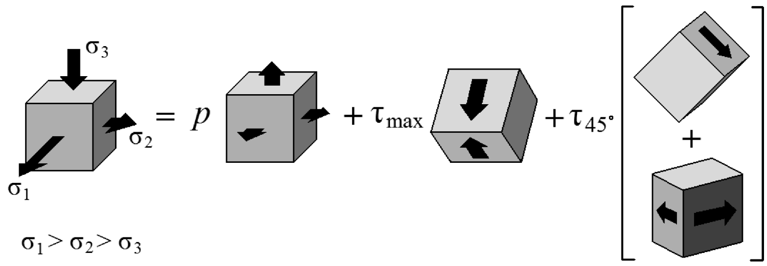

Fig. 1. Stress state decomposition in the basis of eigenstates of the limit state operator.

3. Critical review of the Burzyński criterion

In spite of its generality and clear physical interpretation in the sense of elastic energy, accompanied by mathematical simplicity, one has to note that Burzyński's limit criterion for anisotropic solids is not stated correctly in all of its aspects. General idea of an energy-based criterion with additional function defining contribution of volumetric strain in material effort, is of greatest importance and it emerges to be a simple and effective way to account for i.e. the strength-differential effect in other energy-based hypotheses (see R.B. Pęcherski *et al.* [8], J. Ostrowska–Maciejewska *et al.* [9]). However, there are few misstatements that were not pointed out and discussed in the literature; the main issues which have to be discussed are:

- Principal stress formulation.
- Lack of invariance of the parameters of the criterion.

- Basic coordinate system.
- Anisotropy coefficient λ.
- Isotropy of strength properties.
- Non-unique relation between elastic and strength parameters.

3.1. Principal stress formulation

Burzyński's criterion simplicity is in fact mainly due to its formulation in terms of principal stresses. Practical application of the limit condition (2.11) given by Burzyński (e.g. in numerical computations) enforces the use of the principal stresses directions coordinate system. One should notice that if the parameters of the criterion are to be constant (as it seems to be assumed by Burzyński), one has to assume that (due to anisotropy of the material and especially due to arbitrary orientation of the principal stresses) the whole formulation of the criterion should be invariant with respect to rotations and reflections; it would be isotropic then, what would be an obvious inconsistency. Otherwise, the value of those parameters must change depending on the chosen coordinate system – it is so in case of Burzyński's condition, however this problem was even not mentioned in [1]. It will be discussed in details in the next subsection.

Coordinate system built upon principal stresses directions is not holonomic – local coordinate system at a given point cannot be obtained through differentiation of a position vector along certain curves in the space at that point (especially when inequalities $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ have to be fulfilled), since the stress state distribution changes both in time and space and it may contain singularities or discontinuities.

Stress state determination requires exactly six parameters – six stress state components in any coordinate system or equivalently three stress tensor invariants or principal stresses, and three quantities describing the orientation of the principal stresses directions in the given coordinate system, i.e. three Euler angles or components of the versors indicating directions of principal stresses (nine components with six constraints – three orthogonality conditions and three normalization conditions). Referring only to three parameters, the principal values of the stress tensor does not give us full information about the stress state, which is especially important in case of anisotropic bodies. Simple example should make the problem clear – it is rather obvious that a certain stress state (set of eigenvalues) with its maximal component parallel to the wood fibers, cause much lower material effort than the same set of stresses applied in such a way that the maximal one is perpendicular to the fibers. In case of anisotropic materials, the values of the principal stresses alone are not a sufficient information for the description of the material effort, unless directions of the stresses are fixed. This is also the reason for which any plot of the limit surface for anisotropic bodies in the space of principal stresses, refer in general only to a single, fixed orientation or principal stresses.

3.2. Lack of invariance of stiffness moduli

Stiffness parameters of an anisotropic body (used in criterion) depend strictly on the orientation of a sample referring to the given coordinate system, thus parameters of the criterion must change their values as a result of rotation of the coordinate system, since the directions of principal stresses (which are in fact arbitrary oriented) change – unless these parameters are invariants. It seems that Burzyński might tacitly assume that the parameters of his criterion are constant, which in his energy-based formulation could be possible only if they were invariants. Bulk modulus B as a quantity proportional to a Kelvin modulus of any volumetrically isotropic material is indeed an invariant. Yet all other stiffness moduli used in the criterion, namely L, M, N, which are defined (see relations (2.9)) as a difference between an invariant and a single component of $\mathbf{C}$ (which is not invariant due to anisotropy of $\mathbf{C}$), will in general change their values as the orientation of the coordinate system changes – thus even the name of '*generalized moduli of distortion*' is in fact not strictly correct. Because of lack of invariance of those parameters, whole formulation of the criterion given by Burzyński depends strongly on the choice of coordinate system, which always has to be the principal stresses directions coordinate system. Change of orientation of principal stresses may even cause not only quantitative but also qualitative modification of a yield surface at the given point – i.e. ellipsoidal (brittle materials, closed surface) into paraboloidal (hydrostatic pressure as a safe stress state).

The only solution which seems possible is to consider only the special class of stress states of fixed orientation of principal stresses directions. Yet such constraint is still not sufficient – even in case of coordinate system adapted to the directions of principal stresses and even if the orientation of stresses is fixed (due to e.g. specific use of the element made of the considered material or due to specific way of loading), the coordinate system should be chosen in such way that inequalities $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ will be satisfied. If the values of principal stresses change so that the discussed inequalities in the given coordinate system are no longer true, the coordinate system must be rotated by 90 degrees – in general such rotation is not an element of the symmetry group of arbitrarily chosen anisotropic material, even when volumetrical isotropy is assumed. Burzyński has written clearly that '*current and continued mathematical argument is in present conditions valid only with the assumption of inequality* $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$, [1] – it can be easily shown that (using Burzyński's assumptions) the criterion is not fulfilled in case of uniaxial limit state when one takes: ($\sigma_1 = 0$; $\sigma_2 = k_r$; $\sigma_3 = 0$). This was the way which (with assumption of equalities $\dfrac{M}{L} = \dfrac{M}{N} = 2(1-\lambda)$ which will be discussed below) allowed Burzyński to formulate the condition in such a way that it is indeed fulfilled in case of the limit uniaxial state, what (without those assumptions) is generally not true. Thus the coordinate system (and consistently parameters of the criterion which define the type of a yield surface) change as both orientation or value of principal stresses change.

Rejecting the necessity of fulfilling the system of inequalities $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ (or the inverse one) leads to conclusion that the limit stress in the direction of σ_2 is different than in the directions of two other principal stresses – Eq. (2.11) could be interpreted as a limit condition for the material with anisotropic strength properties for a set of stress states, with fixed principal stresses directions (i.e. parallel to the material symmetry axes). Assuming that k_{ri} and k_{ci} denote tensile and compression strength along the i-th axis respectively ($i = 1, 2, 3$), one can find that neglecting inequalities $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$, the limit condition (2.11) gives us:

$$
\begin{gathered}
k_{r1} = k_{r3} = k_r, \qquad k_{c1} = k_{c3} = k_c, \qquad k_{s2} = \pm k_s,\\
k_{c/r2} = \frac{-k_s^2(\lambda+1)(k_c-k_r)}{2(\lambda+1)k_s^2 + (1-2\lambda)k_ck_r}\\
\pm \frac{k_s\sqrt{(\lambda^2+2\lambda+1)(k_c+k_r)^2k_s^2 - 2(2\lambda^2+\lambda-1)k_c^2k_r^2}}{2(\lambda+1)k_s^2+(1-2\lambda)k_ck_r},\\
k_{s1} = k_{s3} = \pm\sqrt{\frac{2(\lambda+1)}{5-4\lambda}}k_s,
\end{gathered}
\tag{3.1}
$$

what would be suitable for cylindrical or tetragonal symmetry – similar limit criterion for cylindrical symmetry formulated in terms of principal stresses, assuming that their directions are fixed, was analyzed by Theocaris [10]. The above purely mathematical considerations require the expressions under root to be positive. From the expression for $k_{s1} = k_{s3}$ we obtain $\lambda \in (-1; 1.25)$ – this is an interval of possible values of λ for which the above considerations have sense. Furthermore, we require that the term under the root in the formula expressing $k_{c/r2}$ is positive, what leads to the following inequality:

$$
\frac{(\lambda^2+2\lambda+1)}{(2\lambda^2+\lambda-1)} > \frac{2k_c^2k_r^2}{(k_c+k_r)^2k_s^2}.
\tag{3.2}
$$

Physical interpretation of the obtained solutions requires also that the values of tensile and compression strength along x_2 must have different signs, $k_{c2}\cdot k_{r2} < 0$.

Here is a slight inconsistency in the notation in this case, since Burzyński has always considered both k_c and k_r to be positive. However, it does not influence the solution – using well-known Viète's formulas for the product of the roots of the polynomial, we obtain:

$$\frac{(2\lambda - 1)}{(\lambda + 1)} < \frac{2k_s^2}{k_c k_r}. \tag{3.3}$$

However, it has to be emphasized that in the general case, distinguishing of the intermediate stress in Burzyński's formulation must not be mistaken with distinguishing of a certain direction in the material (i.e. as in transversal isotropy), as sometimes it is understood. If a symmetry of the material is described in a given coordinate system ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) and directions of the principal stresses are determined by a set of versors ($\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$) which in general do not correspond with the given coordinate system, then special meaning of intermediate stress is the distinction of $\mathbf{e}'_2 \otimes \mathbf{e}'_2$ (or at most $\mathbf{e}'_2$) having nothing to do with independent of the stress state (thus constant at all points) direction in physical space given by $\mathbf{e}_2$.

3.3. Basic coordinate system

Another inconsistency which has to be discussed is the existence of the basic coordinate system given by a set of equalities (2.7). Its physical interpretation is not quite clear. It is obvious that the coordinate system of principal stresses directions (as well as the one of principal directions of the strain state which in case of anisotropy is not always coaxial with stress state – furthermore, Rychlewski has shown that there exists no such an anisotropic linear elastic material which preserves the coaxiality of stress and strain tensors [11]) is such basic coordinate system – yet it depends on the stress or strain state and thus it is different at each point, what makes it rather impractical in use. Also in case of orthotropy and any other higher symmetry, such basic coordinate system actually exists – axes of such system are parallel to the axes of symmetry of the considered material. Both such systems can be set using simple rotation in physical space, so it is not *'only mathematically possible'*.

However, it is not quite clear if the basic coordinate system really exists in case of lower symmetries (total anisotropy, monoclinic symmetry, trigonal symmetry) independently of the form of stress state – or, speaking in other way, whether there exists such orientation of a coordinate system in physical space, being characteristic for the material (not only for the stress state as in case of principal stresses), which makes it the basic one. There are *'mathematically possible'* rotations in six-dimensional space of symmetric second-order tensors which do not refer to any rotation in physical space, thus there might be no such a real rotation which would satisfy conditions (2.7) for any values of the stress state components. If so, then referring to the compliance tensor components in the relations (2.7) is unnecessary, since the basic coordinate system would be only a stress state-dependent. Actually, even defining such specific coordinate system with the relations (2.7) would be senseless since one always has to take a local coordinate system built upon directions of the principal stresses. Furthermore, if there exists no such a rotation in physical space which would give us the basic coordinate system, then the coordinate system transformation given by (2.7) changes the physical meaning of the components of both the compliance and stress tensor – e.g. components of the stress tensor (appearing in energy density formulation) may emerge to be of an abstract nature - they could not be interpreted as normal or shear stresses. The simplification of the elastic energy density formulation presented by Burzyński might emerge not as general as it first seemed to be and it should be constrained either to the systems of principal stresses direction or one should consider only orthotropy or higher symmetry.

Finding the solution of the problem of existence of the basic coordinate system is equivalent to answering the question if there exists such a basis in physical space in which any compliance tensor $\mathbf{C}$ of a volumetrically isotropic material takes the following form:

$$\mathbf{C} \cong \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{2222} & C_{2233} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & C_{2323} & C_{2331} & C_{2312} \\ & \text{sym} & & & C_{3131} & C_{3112} \\ & & & & & C_{1212} \end{bmatrix}. \tag{3.4}$$

For this very general analysis it is enough to notice that the number of independent components of the compliance tensor of arbitrary symmetry is further decreased from 16 to 10 (please note that the Burzyński's conditions (2.4) still have to be fulfilled) – this indicates that there exist volumetrically isotropic compliance tensors for which there is no such orientation in the physical space, which makes the coordinate system the basic one.

3.4. Anisotropy coefficient λ

As it was shown above in Eq. (2.11), the anisotropy of elastic properties of the considered material was represented by a single parameter λ. It was defined as $\lambda = \dfrac{M^2}{2LN}$. It was also assumed by Burzyński that $\dfrac{M}{L} = \dfrac{M}{N} =$

$2(1-\lambda)$, which is an additional constraint for possible values of parameter λ. It is a consequence of a specific form of the criterion formulation – it can be shown that only if these equalities are true, the criterion is fulfilled in limit uniaxial stress states. Burzyński did not discuss these constraints and stated only that '*it seems reasonable to expect that the interval within which* λ *varies is quite modest, and so that it ranges e.g. from 0 to 1*' [1]. Putting $x = \dfrac{M}{L} = \dfrac{M}{N}$ we obtain $\lambda = \dfrac{M^2}{2LN} = \dfrac{1}{2}x^2$ and finally, substituting both relations in $\dfrac{M}{L} = \dfrac{M}{N} = 2(1-\lambda)$, we obtain the following equation:

$$x^2 + x - 2 = 0. \tag{3.5}$$

There are two roots of the above equation $x_1 = -2$ and $x_2 = 1$. The first one has to be rejected because x was defined as a fraction of two '*stiffness moduli*', which are assumed to be positive. Thus the only result is $x = 1$ which gives us $\lambda = \dfrac{1}{2}$, the value of λ for which the criterion is identical as the criterion for isotropic bodies.

It has to be mentioned that before giving the simplified form of the proposed limit condition (2.11), Burzyński wrote: "[parameters M/N, M/L, M^2/LN] *are not treated* [now] *as representations of the ratio of elasticity constants, but as coefficients particularly connected with the experimental essence of material effort*" [1]. It is not clear how to interpret these words – assuming that in this short sentence Burzyński rejected all previous assumptions on λ (see relations (2.10)), makes all further derivations deprived of theoretical foundation and physical, energy-based interpretation as long as $\lambda \neq \dfrac{1}{2}$. One should remember also that the limit criterion introduced by Burzyński, depends on 4 independent parameters and simple strength tests give us only three values of which the criterion parameters are dependent. Some parameters (e.g anisotropy coefficient λ) must also take into consideration any information about the elastic structure of the material, so they cannot be "*connected*" o n l y "*with the experimental essence of material effort*" – unless there exists a one-to-one correlation between elastic and strength properties of the considered body. This problem is discussed in Subsec. 3.6.

3.5. Isotropy of strength properties

Finally one should also notice that in the whole paper by Burzyński [1] there is no such thing mentioned as anisotropy of *strength properties.* Limit stresses k_c, k_r and k_s are assumed to be independent of the direction of loading.

This makes the criterion to some extent useless since it assumes that each (tensile, compression, shearing) limit stress is the same in any direction, despite the anisotropy of elastic properties of the body.

3.6. Non-unique relation between elastic and strength parameters

Widely known failure criteria formulated by Hill [5] and Hoffman [6] are influenced by their parameters in a linear way. In any such criterion under certain conditions, those parameters can be uniquely expressed in terms of limit stresses. Yield surface can be determined basing only on simple strength tests: uniaxial tension and compression and pure shears in three perpendicular directions. However, the parameters of both mentioned criteria were not interpreted in a strictly physical way. In the contrary to them, most of parameters of Burzyński's criterion (except ω, δ and K) have precise physical meaning and their values can be either directly measured or, at least, estimated through performance of a series of tests and analysis of the obtained elastic constants. They influence the criterion in a linear way, so there might exist one-to-one correlation between them and limit values of stresses. If such relation existed, those parameters could be determined in two ways – by direct measurements of the elastic properties of the body or in a series of simple strength tests. This would indicate that elastic properties of the material determine uniquely its strength properties. Authenticity of such statement should be verified experimentally, however it seems that there might exist two materials of different internal structure, which exhibit macroscopically the same elastic properties but different strength properties (e.g. due to different mechanisms of yielding).

Let us return to the basic form of the failure condition, rejecting later substitutions made by Burzyński. For further simplification, let us assume the we are not considering the cases of symmetries lower than ortothropy, so there exists a fixed coordinate system, independent of the stress state, in which at every point the elastic energy density can be expressed in the form given by Eq. (2.8). Simply substituting (2.8) into (2.2), we obtain:

$$\begin{aligned}(3.6)\qquad \frac{1}{2}B_\omega(\sigma_{11}+\sigma_{22}+\sigma_{33})^2 &+ \frac{1}{2}B_\delta(\sigma_{11}+\sigma_{22}+\sigma_{33}) \\ &+\frac{1}{3}\left[L(\sigma_{22}-\sigma_{33})^2+M(\sigma_{33}-\sigma_{11})^2+N(\sigma_{11}-\sigma_{22})^2\right] \\ &+2P\sigma_{23}^2+2Q\sigma_{31}^2+2R\sigma_{12}^2-K=0,\end{aligned}$$

where $B_\omega = B\omega$, $B_\delta = B\delta$. One can note that if the condition (3.6) is fulfilled for certain values of its parameters, it is also fulfilled if all of them are multiplied by the same constant – this indicates that the relation between those parameters and limit stresses obtained from strength tests cannot be unique. Let us divide (3.6) by K so that we obtain the limit conditions depending on five parameters:

$$\begin{aligned}(3.7)\qquad \frac{1}{2}\widetilde{B_\omega}(\sigma_{11}+\sigma_{22}+\sigma_{33})^2 &+ \frac{1}{2}\widetilde{B_\delta}(\sigma_{11}+\sigma_{22}+\sigma_{33}) \\ &+\frac{1}{3}\Big[\widetilde{L}(\sigma_{22}-\sigma_{33})^2+\widetilde{M}(\sigma_{33}-\sigma_{11})^2+\widetilde{N}(\sigma_{11}-\sigma_{22})^2\Big] \\ &+2\widetilde{P}\sigma_{23}^2+2\widetilde{Q}\sigma_{31}^2+2\widetilde{R}\sigma_{12}^2=-1,\end{aligned}$$

where the parameters with tilde denote the corresponding parameters from (3.6) divided by K.

Assuming pure shear tests, one can easily find

$$\widetilde{P}=\frac{1}{2k_{s1}^2},\qquad \widetilde{Q}=\frac{1}{2k_{s2}^2},\qquad \widetilde{R}=\frac{1}{2k_{s3}^2}.$$

Let us assume that strength properties of the considered body are anisotropic and also that in every direction it exhibits the strength-differential effect. In such case, condition (3.6) has to be fulfilled in six uniaxial states which gives us following overdetermined system of six equations for five parameters of the criterion (contrary to the notation used by Burzyński we assume $k_r > 0$, $k_c < 0$):

$$(3.8)\qquad \frac{1}{3}\begin{bmatrix} 0 & k_{r1}^2 & k_{r1}^2 & \frac{3}{2}k_{r1} & \frac{3}{2}k_{r1}^2 \\ k_{r2}^2 & 0 & k_{r2}^2 & \frac{3}{2}k_{r2} & \frac{3}{2}k_{r2}^2 \\ k_{r3}^2 & k_{r3}^2 & 0 & \frac{3}{2}k_{r3} & \frac{3}{2}k_{r3}^2 \\ 0 & k_{c1}^2 & k_{c1}^2 & \frac{3}{2}k_{c1} & \frac{3}{2}k_{c1}^2 \\ k_{c2}^2 & 0 & k_{c2}^2 & \frac{3}{2}k_{c2} & \frac{3}{2}k_{c2}^2 \\ k_{c3}^2 & k_{c3}^2 & 0 & \frac{3}{2}k_{c3} & \frac{3}{2}k_{c3}^2 \end{bmatrix}\begin{bmatrix}\widetilde{L}\\ \widetilde{M}\\ \widetilde{N}\\ \widetilde{B}_\delta\\ \widetilde{B}_\omega\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}.$$

One can observe that the fifth column of the matrix of coefficients, the one corresponding to the $\widetilde{B}_\omega$ parameter (quadratic pressure influence), can be expressed as a linear combination of the first three columns corresponding to shear moduli $\widetilde{L}$, $\widetilde{M}$, $\widetilde{N}$ – both of two possible 5×5 minors must be then equal to 0 what indicates that the rank of the matrix of coefficients is equal at most to 4. The rank of the augmented matrix is equal to 5, thus it is an inconsistent system of equations and no solution can be found.

4. Proposition of different formulation of the limit criterion

It seems to the author that the final form of the limit condition should depend on Burzyński's stiffness moduli in their unchanged form (their values can be well estimated) and only the parameters K, δ, ω should be determined in a numerical way so that the obtained limit surface fitted the experimental data well. This would give us only three independent parameters which could be used to fit the model to nine independent strength tests. Despite the fact that K, δ, ω are independent of the elastic constants, it is clear that elastic properties would influence the measure of material effort very strongly. Good correlation between the determined model and the experimental results would verify the correctness of Burzyński's hypothesis, in particular the form of the influence function assumed by him. Having determined the limit condition for a sufficiently large set of materials of similar class, may enable finding empirical formulas for the unknown parameters, e.g. $K = K(B, L, M, N, \ldots, k_{r1}, k_{r2}, \ldots, k_{s3})$. In the further analysis, the found formulas for different classes of materials could be compared.

Yet, assuming that parameters K, ω, δ are known as well as the elastic moduli B, L, M, N, P, Q, R, limit stresses can be easily found from the system of Eq. (3.8).

$$
\begin{aligned}
k_{c/r1} &= \frac{-3\widetilde{B}_\delta \pm \sqrt{16(\widetilde{N} + \widetilde{M}) + 24\widetilde{B}_\omega + 9\widetilde{B}_\delta^2}}{4(\widetilde{N} + \widetilde{M}) + 6\widetilde{B}_\omega},\\
k_{c/r2} &= \frac{-3\widetilde{B}_\delta \pm \sqrt{16(\widetilde{N} + \widetilde{L}) + 24\widetilde{B}_\omega + 9\widetilde{B}_\delta^2}}{4(\widetilde{N} + \widetilde{L}) + 6\widetilde{B}_\omega},\\
k_{c/r3} &= \frac{-3\widetilde{B}_\delta \pm \sqrt{16(\widetilde{M} + \widetilde{L}) + 24\widetilde{B}_\omega + 9\widetilde{B}_\delta^2}}{4(\widetilde{M} + \widetilde{L}) + 6\widetilde{B}_\omega},\\
k_{s1} &= \sqrt{\frac{K}{2P}},\\
k_{s2} &= \sqrt{\frac{K}{2Q}},\\
k_{s3} &= \sqrt{\frac{K}{2R}}.
\end{aligned}
\tag{4.1}
$$

The set of acceptable values of parameters K, ω, δ is determined by the following system of inequalities, which are required for the existence of two real solutions k_{ci}, k_{ri} of different signs:

$$\begin{aligned} 16(N+M)+24B\omega+9(B\delta)^2 &> 0,\\ 16(N+L)+24B\omega+9(B\delta)^2 &> 0,\\ 16(L+M)+24B\omega+9(B\delta)^2 &> 0,\\ N+M+\frac{3}{2}B\omega &> 0,\\ N+L+\frac{3}{2}B\omega &> 0,\\ L+M+\frac{3}{2}B\omega &> 0. \end{aligned} \tag{4.2}$$

First three inequalities guarantee positiveness of the expressions under the roots what leads to $k_{ci}, k_{ri} \in \mathbb{R}$ and $k_{ci} \neq k_{ri}$ and last three inequalities are derived using Viète's formulas from the condition $k_{ci} \cdot k_{ri} < 0$ $(i = 1, 2, 3)$. Please note that further constraints for the range of acceptable values of the parameters K, ω, δ can be assumed – e.g. condition of convexity of the limit surface.

5. Summary

It has been shown that failure condition formulation given by Burzyński based on his hypothesis of material effort is not stated correctly in various aspects. However, his original proposition of a hypothesis is of greatest scientific value. It is only the final condition that has to be reformulated. As a concluding remark, it is worth noting that hypothesis of Burzyński distinguishes itself among other similar propositions with certain advantages – it is stated in terms of quantities of clear physical meaning and it enables using large variety of limit surfaces for the description of the limit states for different classes of materials. Great effort made by Burzyński to express the limit condition using possibly small number of parameters, was to make the hypothesis easily applicable in computation; unfortunately it led him to a series of misstatements. However, those inconsistencies do not diminish great importance of the general idea of Burzyński – measure of material effort considered as a combination of independent energy densities, which contribution is determined by a proper stress state – dependent function.

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