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ON FATIGUE STRENGTH UNDER OUT-OF-PHASE SINUSOIDAL LOADINGS

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Let us focus our attention on predicting the influence of phase shifts between sinusoidal stress components of equal frequencies upon the safety factor and the number of cycles-to-failure, in fatigue design of structural members made from structural steels. The general state of stress is considered and analyzed by means of the stress-based calculation procedure developed for the case of in-phase stress components. The basic variable space is divided into the safe region and the failure region. In the failure region the subregion of high-cycle fatigue is determined. Functionals of safety are defined, the minimum values of which can be taken as measures of safety in the safe region and in the failure subregion. It is shown when the phase shift between normal and tangential stress components is most advantageous.

NOTATION

f safety factor in the safe region,
$\widetilde{f}$ functional of safety in the safe region,
$f_x, f_y, \ldots, f_{zx}$ partial safety factors in the safe region,
l limiting factor,
$\widetilde{l}$ instantaneous limiting factor,
$l_x, l_y, \ldots, l_{zx}$ partial limiting factors,
$L_x, L_y, \ldots, L_{zx}$ maximum stress amplitudes corresponding to the highest points of straight regression lines in the plots $\log \sigma_x$ vs. log cycles to failure, $\log \sigma_y$ vs. log cycles to failure, …and $\log \tau_{zx}$ vs. log cycles to failure, respectively,
$m_x, m_y, \ldots, m_{zx}$ exponents in equations of straight regression lines in the aforementioned plots,
n safety factor in the failure subregion,
$\widetilde{n}$ functional of safety in the failure subregion,
$n_x, n_y, \ldots, n_{zx}$ partial safety factors in the failure subregion,
$N_{x0}, N_{y0}, \ldots, N_{zx0}$ numbers of stress cycles to failure corresponding to the intercepts of straight lines in the aforementioned plots,
N_0 required number of cycles to achieve a given design life,
N number of stress cycles to failure under combined load,
t time,

Z_{g0} fatigue limit for alternate bending,
Z_{rc} fatigue limit for tension-compression,
Z_{rj} fatigue limit for pulsating tension,
Z_{sj} fatigue limit for pulsating torsion,
Z_{s0} fatigue limit for twisting,
$Z_x, Z_y, \ldots, Z_{zx}$ fatigue limits for simple states of stress $\widetilde{\sigma}_x, \widetilde{\sigma}_y, \ldots, \widetilde{\tau}_{zx}$, respectively,
$\alpha_x, \alpha_y, \ldots, \alpha_{zx}$ phase angles of stress components $\widetilde{\sigma}_x, \widetilde{\sigma}_y \ldots$ and $\widetilde{\tau}_{zx}$, respectively,
$\widetilde{\sigma}_x, \widetilde{\sigma}_y, \ldots, \widetilde{\tau}_{zx}$ Cartesian stress components,
$\sigma_x, \sigma_y, \ldots, \tau_{zx}$ amplitudes of the Cartesian stress components,
ω angular frequency.

1. Introduction

The influence of phase shifts between sinusoidal stress components of equal frequencies on fatigue lifetime and crack propagation was observed in various tests under combined loads (e.g. [1–3]). For example, in-phase and out-of-phase bending and torsion, having equal principal stress ranges, resulted in experimental fatigue lifes of tube-to-plate weldments differing by an order of magnitude [3]. Another combined load was analyzed in [4], where the relationship between the safety factor and phase angles in the case of superposition of two sinusoidal normal stress components of equal frequencies was derived. As pointed out, the safety factor under out-of-phase bending and tension-compression is by 41% greater than that under such loadings in phase.

In the literature analogous relationships for more complex cases of combined loads do not exist. The reasons for that may be the scatter inherent in fatigue tests and intricacy of fatigue behaviour, as well as the fact that natural processes produce random rather than sinusoidal excitations, which makes it difficult to assess the significance of some factors that may influence the cyclic life.

However, in certain technological processes the phase shifts between loadings are imposed by the designer, which substantiates an effort to determine the relevant interdependencies. For this purpose, in the present paper an application of the formulae derived in [5,6] on the basis of Huber–Mises distortion-energy theory and linear logarithmic $S-N$ curves is considered.

The distortion energy theory is valid for ductile materials and is widely accepted in static problems. In fatigue analysis, a number of theories exist as to the actual mechanisms of crack initiation: however, there is no one mechanism that could be universally accepted [7]. Although some experimental data indicate applicability of the distortion energy theory to the fatigue life prediction [8], other test results may lead to the opposite conclusion [3]. Therefore, in [5,6] an adaptation of this theory for elastic applications in fatigue design of structural members made from structural steels was proposed, under the assumption that sinusoidal stress components are in phase. In the following, this assumption will be avoided.

2. Safety factors and fatigue lifetime under in-phase sinusoidal loadings

In fatigue design, the basic variable space can be divided into a safe region and a failure region. For the stress-based fatigue calculations a failure subregion can be determined [6]. The equation of boundary surface (failure surface) between the safe region and the failure subregion is

$$f^{-2} - 1 = 0, \tag{2.1}$$

where f is the safety factor in the safe region defined for in-phase stress components as [5]

$$f = \Big[f_x^{-2} + f_y^{-2} + f_z^{-2} - (f_x f_y)^{-1} - (f_y f_z)^{-1} - (f_z f_x)^{-1} + f_{xy}^{-2} + f_{yz}^{-2} + f_{zx}^{-2}\Big]^{-1/2}, \tag{2.2}$$

where

$$f_x = \frac{Z_x}{\sigma_x}, \quad f_y = \frac{Z_y}{\sigma_y}, \quad f_z = \frac{Z_z}{\sigma_z}, \qquad f_{xy} = \frac{Z_{xy}}{\tau_{xy}}, \quad f_{yz} = \frac{Z_{yz}}{\tau_{yz}}, \quad f_{zx} = \frac{Z_{zx}}{\tau_{zx}} \tag{2.3}$$

are the partial safety factors in the safe region, $\sigma_x, \sigma_y, \ldots, \tau_{zx}$ are the amplitudes of Cartesian stress components at a given point of a structural member, and $Z_x, Z_y, \ldots, Z_{zx}$ are the respective fatigue limits determined for each simple state of stress separately. For instance, in a complex state of stress with in-phase sinusoidal components of amplitudes $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}$ and τ_{yz} produced by tension-compression, pulsating tension, alternate bending, twisting and pulsating torsion, respectively, and $\tau_{zx} = 0$, the following quantities

$$Z_x = Z_{rc}, \quad Z_y = Z_{rj}, \quad Z_z = Z_{g0}, \quad Z_{xy} = Z_{s0}, \quad Z_{yz} = Z_{sj}, \quad \tau_{zx} = 0$$

are to be substituted into Eqs. (2.3), where

Z_{rc} fatigue limit for tension-compression,

Z_{rj} fatigue limit for pulsating tension,

Z_{g0} fatigue limit for alternate bending,

Z_{s0} fatigue limit for twisting,

Z_{sj} fatigue limit for pulsating torsion.

The case $f < 1$ corresponds to the failure region and $f \geq 1$ to the safe region. Safety analysis in both these regions must be based on different relationships. Equations (2.2) and (2.3) are convenient in the safe region, whereas in the failure region a safety factor related to fatigue life is more appropriate, e.g.

$$n = \frac{N}{N_0}, \tag{2.4}$$

where N is the number of stress cycles to cause failure under combined load, and N_0 is the required number of stress cycles to achieve a given design life. Such a factor was determined in [6], for the failure suberegion only, as

$$n = \Big[n_x^{-2} + n_y^{-2} + n_z^{-2} - (n_x n_y)^{-1} - (n_y n_z)^{-1} - (n_z n_x)^{-1} + n_{xy}^{-2} + n_{yz}^{-2} + n_{zx}^{-2}\Big]^{-1/2}, \tag{2.5}$$

where

$$\begin{aligned} n_x &= \frac{N_{x0}}{N_0}\left(\frac{Z_x}{\sigma_x}\right)^{m_x}, & n_y &= \frac{N_{y0}}{N_0}\left(\frac{Z_y}{\sigma_y}\right)^{m_y}, \\ n_z &= \frac{N_{z0}}{N_0}\left(\frac{Z_z}{\sigma_z}\right)^{m_z}, & n_{xy} &= \frac{N_{xy0}}{N_0}\left(\frac{Z_{xy}}{\tau_{xy}}\right)^{m_{xy}}, \\ n_{yz} &= \frac{N_{yz0}}{N_0}\left(\frac{Z_{yz}}{\tau_{yz}}\right)^{m_{yz}}, & n_{zx} &= \frac{N_{zx0}}{N_0}\left(\frac{Z_{zx}}{\tau_{zx}}\right)^{m_{zx}} \end{aligned} \tag{2.6}$$

are the partial safety factors in the failure subregion, N_{x0} is the number of stress cycles to failure corresponding to the intercept of straight lines in the plot $\log\sigma_x$ vs. log cycles to failure, and m_x is the exponent in equation of the regression line in the same plot. $N_{y0}, N_{z0}, \ldots, N_{zx0}$ and $m_y, m_z, \ldots, m_{zx}$ are defined similarly.

Since Eq. (2.5) is valid in the failure subregion only, its outer boundary should be determined as well. According to [6], equation of the outer boundary surface of the failure subregion reads

$$l^{-2} - 1 = 0. \tag{2.7}$$

In Eq. (2.7), l is the limiting factor defined for in-phase stress components as

$$(2.8)\qquad l = \Big[l_x^{-2} + l_y^{-2} + l_z^{-2} - (l_x l_y)^{-1} - (l_y l_z)^{-1} - (l_z l_x)^{-1} + l_{xy}^{-2} + l_{yz}^{-2} + l_{zx}^{-2}\Big]^{-1/2},$$

where

$$(2.9)\qquad \begin{aligned} l_x &= \frac{L_x}{\sigma_x}, & l_y &= \frac{L_y}{\sigma_y}, & l_z &= \frac{L_z}{\sigma_z}, \\ l_{xy} &= \frac{L_{xy}}{\tau_{xy}}, & l_{yz} &= \frac{L_{yz}}{\tau_{yz}}, & l_{zx} &= \frac{L_{zx}}{\tau_{zx}} \end{aligned}$$

are the partial limiting factors. L_x is the maximum stress amplitude σ_x that satisfies the equation of the straight regression line in the plot $\log \sigma_x$ vs. log cycles to failure. $L_y, L_z, \ldots, L_{zx}$ are defined analogously. The failure subregion is exceeded when $l < 1$.

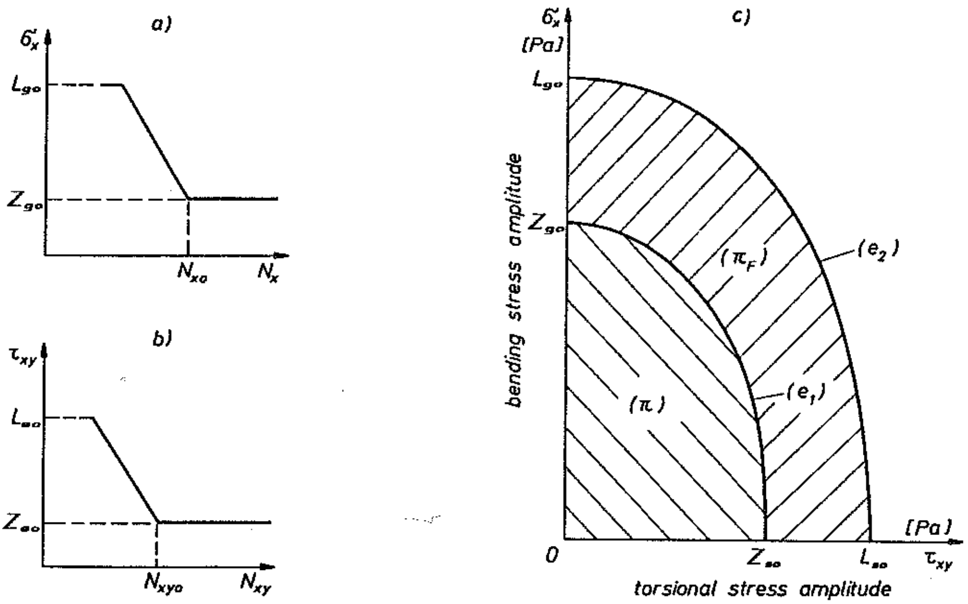

FIG. 1. Log stress amplitude vs. log cycles to failure for alternate bending (a) and twisting (b). (c) – safe region (π) and failure subregion (π_F) for combined in-phase bending and twisting (in linear scale).

As an example, in Fig. 1c the safe region and the failure subregion for in-phase sinusoidal stress components of amplitudes σ_x and τ_{xy} produced by alternate bending and twisting are illustrated. The corresponding $S-N$ curves are presented in Fig. 1a and Fig. 1b. In this case $Z_x = Z_{g0}$, $L_x = L_{g0}$,

$Z_{xy} = Z_{s0}$ and $L_{xy} = L_{s0}$. The boundary between the safe region and the failure subregion, and the outer boundary of the latter, are denoted (e_1) and (e_2), respectively. The equation of curve (e_1) (interaction equation) reads

$$\left(\frac{\sigma_x}{Z_{g0}}\right)^2 + \left(\frac{\tau_{xy}}{Z_{s0}}\right)^2 = 1, \tag{2.10}$$

whereas that of curve (e_2) is

$$\left(\frac{\sigma_x}{L_{g0}}\right)^2 + \left(\frac{\tau_{xy}}{L_{s0}}\right)^2 = 1. \tag{2.11}$$

3. Safety factors and fatigue lifetime under out-of-phase sinusoidal loadings

In the case of in-phase sinusoidal loadings, Cartesian stress components are

$$\widetilde{\sigma}_x = \sigma_x \sin\omega t, \qquad \widetilde{\sigma}_y = \sigma_y \sin\omega t, \ldots, \qquad \widetilde{\tau}_{zx} = \tau_{zx} \sin\omega t. \tag{3.1}$$

According to Sec. 2 and Eqs. (3.1), it is possible to determine the instantaneous safety factor in the safe region

$$\widetilde{f} = \Big[\widetilde{f}_x^{-2} + \widetilde{f}_y^{-2} + \widetilde{f}_z^{-2} - \left(\widetilde{f}_x\widetilde{f}_y\right)^{-1} - \left(\widetilde{f}_y\widetilde{f}_z\right)^{-1} - \left(\widetilde{f}_z\widetilde{f}_x\right)^{-1} + \widetilde{f}_{xy}^{-2} + \widetilde{f}_{yz}^{-2} + \widetilde{f}_{zx}^{-2}\Big]^{-1/2}, \tag{3.2}$$

the instantaneous safety factor in the failure subregion

$$\widetilde{n} = \Big[\widetilde{n}_x^{-2} + \widetilde{n}_y^{-2} + \widetilde{n}_z^{-2} - (\widetilde{n}_x\widetilde{n}_y)^{-1} - (\widetilde{n}_y\widetilde{n}_x)^{-1} - (\widetilde{n}_z\widetilde{n}_x)^{-1} + \widetilde{n}_{xy}^{-2} + \widetilde{n}_{yz}^{-2} + \widetilde{n}_{zx}^{-2}\Big]^{-1/2}, \tag{3.3}$$

and the instantaneous limiting factor

$$\widetilde{l} = \Big[\widetilde{l}_x^{-2} + \widetilde{l}_y^{-2} + \widetilde{l}_z^{-2} - \left(\widetilde{l}_x\widetilde{l}_y\right)^{-1} - \left(\widetilde{l}_y\widetilde{l}_z\right)^{-1} - \left(\widetilde{l}_z\widetilde{l}_x\right)^{-1} + \widetilde{l}_{xy}^{-2} + \widetilde{l}_{yz}^{-2} + \widetilde{l}_{zx}^{-2}\Big]^{-1/2}, \tag{3.4}$$

where

$$(3.5)\qquad \widetilde{f}_x = \frac{Z_x}{\widetilde{\sigma}_x}, \qquad \widetilde{f}_y = \frac{Z_y}{\widetilde{\sigma}_y}, \ \ldots, \qquad \widetilde{f}_{zx} = \frac{Z_{zx}}{\widetilde{\tau}_{zx}},$$

$$(3.6)\qquad \begin{aligned} \widetilde{n}_x &= \frac{N_{x0}}{N_0}\left(\frac{Z_x}{\widetilde{\sigma}_x}\right)^{m_x}, \qquad \widetilde{n}_y = \frac{N_{y0}}{N_0}\left(\frac{Z_y}{\widetilde{\sigma}_y}\right)^{m_y}, \ \ldots, \\ \widetilde{n}_{zx} &= \frac{N_{zx0}}{N_0}\left(\frac{Z_{zx}}{\widetilde{\tau}_{zx}}\right)^{m_{zx}}, \end{aligned}$$

$$(3.7)\qquad \widetilde{l}_x = \frac{L_x}{\widetilde{\sigma}_x}, \qquad \widetilde{l}_y = \frac{L_y}{\widetilde{\sigma}_y}, \ \ldots, \qquad \widetilde{l}_{zx} = \frac{L_{zx}}{\widetilde{\tau}_{zx}}.$$

From Eqs. (2.2), (2.3), (2.5), (2.6), (2.8), (2.9) and (3.1)–(3.7) it follows that the instantaneous factors range from

$$(3.8)\qquad \widetilde{f}_{\min} = f,$$

$$(3.9)\qquad \widetilde{n}_{\min} = n,$$

$$(3.10)\qquad \widetilde{l}_{\min} = l$$

at $\sin\omega t = \pm 1$, to infinity at $\sin\omega t = 0$. It means that the factors f, n and l can be regarded as the minimum values of the instantaneous factors $\widetilde{f}$, $\widetilde{n}$ and $\widetilde{l}$ in the time domain. Hence it may be concluded that

1) the instantaneous safety factor $\widetilde{f}$ can be regarded as the functional of safety in the safe region and its minimum value can be taken as a measure of safety in this region when $\widetilde{f}_{\min} \geq 1$;

and that

2) the instantaneous safety factor $\widetilde{n}$ can be regarded as the functional of safety in the failure subregion and its minimum value can be taken as a measure of safety in this subregion when $\widetilde{f}_{\min} < 1 \leq \widetilde{l}_{\min}$.

Consequently, in order to assess the influence of phase angles $\alpha_x, \alpha_y, \ldots, \alpha_{zx}$ on the safety of a structural member subjected to the stress with components

$$(3.11)\qquad \begin{aligned} \widetilde{\sigma}_x &= \sigma_x \sin(\omega t + \alpha_x), \\ \widetilde{\sigma}_y &= \sigma_y \sin(\omega t + \alpha_y), \ \ldots, \\ \widetilde{\tau}_{zx} &= \tau_{zx} \sin(\omega t + \alpha_{zx}), \end{aligned}$$

the functional of safety in the safe region, determined by Eqs. (3.2), (3.5) and (3.11), must be analyzed as a function of time and its minimum value calculated. Is this value not smaller than unity, the calculations will thus be completed. In the opposite case, one has to verify if the outer boundary surface of the failure subregion is not exceeded. For this purpose, Eqs. (3.7) and (3.11) must be substituted into Eq. (3.4) and the minimum value $\widetilde{l}_{\min}$ of the instantaneous limiting factor must be determined. Should it happen that $\widetilde{l}_{\min} < 1$, the presented calculation procedure cannot be applied. In the case of $\widetilde{l}_{\min} \geq 1$, the number of cycles to failure can be, according to Eqs. (2.4), (3.3), (3.6), (3.9) and (3.11), evaluated as

$$N = \widetilde{n}_{\min} N_0. \tag{3.12}$$

4. Example

As an example, combined alternate bending and twisting of equal frequencies is considered. In this case, the state of stress is determined by the components

$$\widetilde{\sigma}_x = \sigma_x \sin \omega t, \qquad \widetilde{\tau}_{xy} = \tau_{xy} \sin(\omega t + \alpha). \tag{4.1}$$

According to Secs. 2 and 3, the influence of the phase angle α on the safety factor can be analyzed either in the safe region, i.e. when the condition

$$\left\{ \left[\left(\frac{\widetilde{\sigma}_x}{Z_{g0}} \right)^2 + \left(\frac{\widetilde{\tau}_{xy}}{Z_{s0}} \right)^2 \right]^{-1/2} \right\}_{\min} \geq 1 \tag{4.2}$$

is fulfilled, or in the failure subregion, i.e. when the inequalities

$$\left\{ \left[\left(\frac{\widetilde{\sigma}_x}{Z_{g0}} \right)^2 + \left(\frac{\widetilde{\tau}_{xy}}{Z_{s0}} \right)^2 \right]^{-1/2} \right\}_{\min} < 1 \leq \left\{ \left[\left(\frac{\widetilde{\sigma}_x}{L_{g0}} \right)^2 + \left(\frac{\widetilde{\tau}_{xy}}{L_{s0}} \right)^2 \right]^{-1/2} \right\}_{\min} \tag{4.3}$$

are fulfilled. The analyzed effect can be presented as the minimum value of the quotient

$$\frac{\widetilde{f}}{f} = \left(\frac{f_x^{-2} + f_{xy}^{-2}}{\widetilde{f}_x^{-2} + \widetilde{f}_{xy}^{-2}} \right)^{1/2} \tag{4.4}$$

in the safe region, and as the minimum value of the quotient

$$\frac{\widetilde{n}}{n} = \left(\frac{n_x^{-2} + n_{xy}^{-2}}{\widetilde{n}_x^{-2} + \widetilde{n}_{xy}^{-2}} \right)^{1/2} \tag{4.5}$$

in the failure subregion. On the basis of Eqs. (2.3), (2.6), (3.5), (3.6) and (4.1), one gets

$$\frac{\widetilde{f}}{f} = \left[\frac{f_x^{-2} + f_{xy}^{-2}}{f_x^{-2} \sin^2 \omega t + f_{xy}^{-2} \sin^2(\omega t + \alpha)} \right]^{1/2} \tag{4.6}$$

in the safe region, and

$$(4.7) \qquad \frac{\tilde{n}}{n} = \left\{ \frac{n_x^{-2} + n_{xy}^{-2}}{n_x^{-2}(\sin \omega t)^{2m_x} + n_{xy}^{-2}[\sin(\omega t + \alpha)]^{2m_{xy}}} \right\}^{1/2}$$

in the failure subregion. The results of calculations are depicted in Fig. 2. The minimum values of $\tilde{f}/f$ and $\tilde{n}/n$ presented in Fig. 2 were found within the time period $t \in [0, \pi/2\omega]$ (in subsequent periods the results are the same).

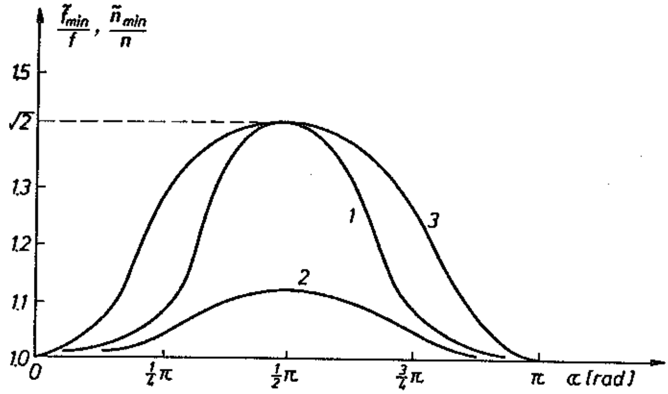

FIG. 2. Influence of the phase shift between a normal and tangential stress components on the safety factor in the safe region (curves *1* and *2*) and in the failure subregion (curve *3*) for 1. $f_x/f_{xy} = 1$, 2. $f_x/f_{xy} = 0.5$ and 2, 3. $n_x/n_{xy} = 1$, $m_x = m_{xy} = 3$.

5. Concluding remarks

From the above example it follows that the phase shift between normal and tangential stress components is advantageous. It means that either the safety factor becomes greater, or greater stress component amplitudes can be applied. The smaller is the difference in values of the partial safety factors, and the closer to $\pi/2$ is the phase shift, the better is the effect.

Eqs. (4.2)–(4.7) correspond to the elliptical interaction equation (2.10). In order to account for other forms of interaction equations of normal and tangential stress components, the similar approach can be employed.

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