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INTERACTION OF MEMBRANE AND BENDING FORCES IN PLATES AT NONLINEAR VIBRATIONS

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The subject of this paper is the analysis of the dynamic response of circular and rectangular elastic-viscoplastic plates subjected to impulsive loading. The range of what is referred to as moderately large deflections has been considered. All the displacement components and their time derivatives in the description of the kinematics are included. The equations of motion have been formulated by using the principle of virtual work. Approximate solutions are found by the orthogonalization principle. As regards the numerical aspects of this method, they are characterized by the necessity of applying typical techniques of numerical integration of the equations of motion and the constitutive relations. Principal emphasis is laid on the study of the co-operation between membrane and bending forces over a wide range of plate deflection. The influence of such factors as the type of edge support, the character of dynamic load and the kind of material of the plate have been discussed. It has been found that plates on hinged supports are more sensitive to membrane effects than those with clamped edges. Intensive dynamic loads with long periods of action lead in most cases to membrane-type mechanisms of deformation. Initial velocity pulses make the plate move with a strongly non-stationary displacement velocity field, the deformation being characterized by a flexural way of producing deformation.

1. INTRODUCTION

The process of deformation of a plate subjected to a pulse load differs essentially from a process caused by a static load. In the case of dynamic load an important role is played by the inertia effects, and the motion of the plate proceeds with a variable, non-stationary displacement velocity. The rate of variation of transient forms of motion is influenced above all by the space-time character of the load and its intensity. In addition, if the plate undergoes large dynamic deformation, shell-type states of strain, not necessarily tensile, are produced.

A precise description of the dynamic process requires giving up some of the approximate assumptions, commonly used in the analysis of dynamic problems of inelastic structures. The character of the model assumed to describe the behaviour of the material of the plate is very important.

Rigid-plastic models ensure, in general, rapid and effective appraisal of permanent deflection, the time to failure, etc. They do not enable us, however, to make a correct description of the phenomena occurring in the initial phase of motion of the plate and the vibrations occurring immediately after the deflection amplitude has reached its maximum.

Considerable variety of forms of motion in the initial phase of pure elastic deformation may give rise to high compression forces of plate-type. Moreover, the initially elastic reaction of the plate is essential for the mechanism of formation of regions of active plastic flow. This mechanism is very much different from that in the analogous case of a rigid-plastic plate.

The kinematic description of the problem is also of fundamental importance. The usual simplification which consists in assuming the transverse displacement component as the only one to be considered in the kinematic analysis, may lead to the strain distribution in the plate not conformable to the exprimental results. In the case of large deformation geometrical nonlinearities must also be considered. They are decided upon by the form of the strain-displacement relations. In a wide class of plate problems it suffices to take into consideration the nonlinear components resulting from the considerable rotations of plate elements out of the plane of the plate. A critical discussion of the assumptions, methods and results obtained within the framework of dynamics of inelastic plates can be found in the survey works of JONES [1], WIERZBICKI [2], and NURICK and MARTIN [3,4].

In the case of geometrically and physically nonlinear problems typical numerical methods of finite elements or differences are effective. The monograph of KLEIBER and WOŹNIAK [5], the works of ARGYRIS *et al.* [6] and LEECH *et al.* [7] are illustrative for the theory and application problem of numerical techniques in the domain of nonlinear analysis of structures.

In the present paper the analytical-numerical approach of BĄK and DORNOWSKI [8,9] to problems of dynamics of elastic-viscoplastic plates undergoing flexural deformation considered to be moderately large will be used. A principal feature of that method is integral formulation based on the principle of virtual work as applied to the initial undeformed configuration of the plate. Numerical aspects of that solution method reduce to the fact that typical numerical integration techniques are needed for the equations of motion and the constitutive relations.

Circular and rectangular elastic-viscoplastic plates clamped or resting on hinged supports will be analysed in detail. The loads will be those of pressure pulse or initial velocity pulse, as applied to a portion or the entire surface of the plate. The kinematic description will take into account all the displacement components and time derivatives. Principal emphasis will be laid on the interaction between membrane forces and bending moments over a wide range of plate deflection. The description of the mechanism of motion will be uniform during the entire deformation process. With such an approach no deformation phase will be treated as privileged beforehand.

2. Theoretical considerations

The fundamental assumptions of the von Kármán theory of moderately large deflections of thin elastic plates will be used. In the case of statical problems this theory is presented in the textbook of Fung [10] and the monographs of Kączkowski [11] and Wolmir [12]. Nonlinear problems of elastic plates and an analogy of the von Kármán theory for problems of dynamics (only transversal inertia being considered) are discussed in the monograph of Chia [13]. Having in view the analysis of interaction of vibrations in the plane of the plate with transversal vibrations under conditions of inelastic deformation, some generalizations will be necessary.

Let us assume that a plate of uniform thicknes H occupies, at the initial instant of time, a region Ω of the physical space. Let us now considere in that space a Cartesian reference frame $\{x_\alpha, z\}$, $\alpha = 1, 2$ in such a manner that the axes x_α lie in the middle plane Ω^* of the undeformed plate. Then the region $\Omega = \Omega^* \times < -H/2, H/2 >$ is also a region of variation of the material coordinates $X_\Delta \in \Omega^*$, $\Delta = 1, 2$, $Z \in < -H/2, H/2 >$ and will be termed the initial configuration of the plate. Let us denote by T the time interval within which the motion of the plate is to be considered. The time coordinate t corresponds to the time of deformation, $t \in T$. All the quantities considered will be referred to the initial configuration, the material (Lagrange) description being used. Symbols with lower indices, in the form of Greek capital letters will be used. These indices take values 1 or 2. The summation operation will be performed without using the summation sign, if an index occurs twice. Partial derivatives will be denoted by a comma. Thus, for instance, $\partial(\cdot)/\partial X_\Delta = (\cdot)_{,\Delta}$.

By describing the plate by means of a displacement field we can characterize the value of the deformation considered. We analyze, therefore, a wide class of problems of dynamics of plates, in which the amplitudes of deflection of the plate are of the order of plate thickness, and the amplitudes of in-plane displacement may be considered to be small. Such a deformation is accompanied by a tensor of displacement gradients, the symmetric part of which (the strain) being a quantity by one order of magnitude lower than the asymmetric (rotation) part and small as compared with unity.

The assumptions just mentioned determine the order of geometrical non-linearity in the description of the kinematics of the problem. This description is completed by the following relations resulting from the Kirchhoff - Love assumption on straight normal lines:

$$E_{\Delta\Lambda}(X_\Gamma, Z, t) = L_{\Delta\Lambda}(X_\Gamma, t) + ZK_{\Delta\Lambda}(X_\Gamma, t), \tag{2.1}$$

where $E_{\Delta\Lambda}$ denotes the components of the Green displacement strain tensor with the approximation assumed in the form of finite deformation. The generalized deformation tensors

$$\begin{aligned} L_{\Delta\Lambda}(X_\Gamma, t) &= \frac{1}{2}[U_{\Delta\Lambda}(X_\Gamma, t) + U_{\Lambda\Delta}(X_\Gamma, t) \\ &\qquad + W_{,\Delta}(X_\Gamma, t)W_{,\Lambda}(X_\Gamma, t)], \\ K_{\Delta\Lambda}(X_\Gamma, t) &= -W_{,\Delta\Lambda}(X_\Gamma, t) \end{aligned} \tag{2.2}$$

describe the state of deformation and curvature of the middle surface of the deformed plate. The symbols U_Δ, W in the above relations denote the components of the displacement vector field in the middle plane. Those components are associated of the relevant directions of the material axes X_Δ and Z.

The generalized stresses (internal forces in a plate) are defined on the basis of the relevant (an appropriate) measure of the state of stress. Such a measure is the tensor $S_{\Delta\Lambda}$, which is associated with the assumed tensor $E_{\Delta\Lambda}$ in the sense of the principle of virtual work. In view of the limitations of kinematic nature imposed on the deformation, the measure $S_{\Delta\Lambda}$ may be treated as an approximation to the second Piola - Kirchhoff stress tensor. For a shell state the internal forces in the plate are determined by the following relations:

$$\begin{aligned} N_{\Delta\Lambda}(X_\Gamma, t) &= \int_{-H/2}^{H/2} S_{\Delta\Lambda}(X_\Gamma, Z, t)\, dZ, \\ M_{\Delta\Lambda}(X_\Gamma, t) &= \int_{-H/2}^{H/2} S_{\Delta\Lambda}(X_\Gamma, Z, t) Z\, dZ, \\ Q_\Delta(X_\Gamma, t) &= \int_{-H/2}^{H/2} S_{\Delta Z}(X_\Gamma, Z, t)\, dZ. \end{aligned} \tag{2.3}$$

Because $E_{\Delta Z} = 0$, the shear forces Q_Δ have a character of forces which are passive in the energy sense.

Equations of dynamic equilibrium are established by giving up local approach and using an approximate orthogonalization method, the theoretical foundations of which have been explained in [8]. This method may be treated as a generalization of the method of decomposition into series of eigenfunctions of classical problems of elasticity. The eigenforms of linear elastic vibrations loose some of their properties as a result of the changes in the geometry of the deformed plate becoming plastic. In further nonlinear analysis they are still, however a set of orthogonal functions and satisfy the kinematic boundary conditions assumed. It should be observed that in the case of a physically nonlinear dynamic problem of plates there exists no set of complete orthogonal basic functions of the solution, and the principle of superposition is not valid either. The accuracy of the approximate solution based on the eigenfunctions of the corresponding linear problem have been verified in [9] by confrontation with the experimental results available.

The components of the displacement field are assumed in the form of the following combinations

$$
\begin{aligned}
U_\Delta(X_\Lambda, t) &= \sum_{n=1}^{N} U_{\Delta n}(t)\chi_{\Delta n}(X_\Lambda), \\
W(X_\Lambda, t) &= \sum_{n=1}^{N} W_n(t)\Psi_n(X_\Lambda).
\end{aligned} \tag{2.4}
$$

The basic functions $\chi_{\Delta n}(X_\Lambda)$, $\Psi_n(X_\Lambda)$ are forms of natural vibrations of the corresponding linear problems.

By treating the principle of virtual works formulated in the material description as an orthogonalization principle, we obtain the following set of equations of dynamic equilibrium of the plate

$$
\begin{aligned}
\ddot{U}_{\Delta n} &= -\frac{1}{\mu \int\limits_{\Omega^*} \chi^2_{\Delta n} d\Omega^*} \int\limits_{\Omega^*} N_{\Delta\Lambda} \chi_{\Delta n,\Lambda} d\Omega^*, \qquad \Delta \text{ not summed}, \\
\ddot{W}_n &= \frac{P_n}{\mu} - \frac{1}{\mu \int\limits_{\Omega^*} \Psi_n^2 d\Omega^*} \int\limits_{\Omega^*} \left(W_{,\Delta} N_{\Delta\Lambda} \Psi_{n,\Lambda} - M_{\Delta\Lambda} \Psi_{n,\Delta\Lambda}\right) d\Omega^*.
\end{aligned} \tag{2.5}
$$

The dots in the above ordinary differential equations denote material differentiation with respect to the time t, and $\mu = \rho H$ is the mass per unit area of the region Ω^*. The forces of rotational inertia have been neglected and, in agreement with the kinematic assumptions, the principle of solidification has ben retained only as regards the equilibrium condition $(2.5)_1$. The influence of the effects of geometrical nonlinearities is seen in Eq. $(2.5)_2$.

The equations of motion are coupled through the forces $N_{\Delta\Lambda}$. The external excitation component has the form

$$(2.6)\qquad P_n(t) = \frac{\int\limits_{\Omega^*} p(X_\Lambda, t)\Psi_n(X_\Lambda)\, d\Omega^*}{\int\limits_{\Omega^*} \Psi_n^2(X_\Lambda)\, d\Omega^*},$$

where $p(X_\Lambda, t)$ is the dynamic transverse surface load. In the case of a transverse loading by an initial velocity pulse $V_0(X_\Lambda)$, Eq. $(2.5)_2$ is homogeneous the information on excitation being contained in the following initial conditions of the problem

$$(2.7)\qquad \begin{aligned} &U_{\Delta n}(0) = \dot{U}_{\Delta n}(0) = W_n(0) = 0, \\ &\dot{W}_n(0) = \frac{\int\limits_{\Omega^*} V_0(X_\Lambda)\Psi_n(X_\Lambda)\, d\Omega^*}{\int\limits_{\Omega^*} \Psi_n^2(X_\Lambda)\, d\Omega^*}. \end{aligned}$$

For a plate subjected to a load $p(X_\Lambda, t)$ we assume the zero homogeneous initial boundary conditions

$$(2.8)\qquad U_{\Delta n}(0) = \dot{U}_{\Delta n}(0) = W_n(0) = \dot{W}_n(0) = 0.$$

The remaining group of equations describing the problem of plate dynamics are physical relations. The deformation properties of a metal under dynamic load are well expressed by the constitutive relations of perfect elasticity-viscoplasticity as formulated by Perzyna for infinitesimal deformations, [14]. This law will be used within the range of strain produced by the accompanying moderately large deflections. They will be expressed by using measures of strain and stress suitable for the material description assumed, structure of the latter being preserved. Thus, the following additivity law is assumed for the strain rate tensor:

$$(2.9)\qquad \dot{E}_{\Delta\Lambda} = \dot{E}^{e}_{\Delta\Lambda} + \dot{E}^{p}_{\Delta\Lambda}.$$

The elastic part of the strain rate tensor $\dot{E}^{e}_{\Delta\Lambda}$ will be determined from the linear law of elasticity. For a plane state of stress the following equations are valid within the range of elastic strains

$$(2.10)\qquad \begin{aligned} \dot{E}^{e}_{\Delta\Lambda} &= \frac{1}{E}\left[(1+\nu)\,\dot{S}_{\Delta\Lambda} - \nu\,\dot{S}_{\Gamma\Gamma}\,\delta_{\Delta\Lambda}\right], \\ \dot{S}_{\Delta\Lambda} &= \frac{E}{1-\nu^2}\left[(1-\nu)\,\dot{E}^{e}_{\Delta\Lambda} + \nu\,\dot{E}^{e}_{\Gamma\Gamma}\delta_{\Delta\Lambda}\right]. \end{aligned}$$

The constants E and ν denote Young's modulus and the Poisson ratio, respectively. The material undergoes plastic strains, if the static yield condition is reached

(2.11) $$F = \frac{\sqrt{J_2}}{k} - 1,$$

where

(2.12) $$J_2 = \frac{1}{6}\left(3S_{\Delta\Lambda}S_{\Delta\Lambda} - S_{\Delta\Delta}S_{\Lambda\Lambda}\right)$$

in the second invariant of the strain deviator and k denotes the yield point at pure shear.

The relations describing, the inelastic properties of the material can be expressed thus by:

(2.13) $$\dot{E}^{\,p}_{\Delta\Lambda} = \dot{\lambda}\left(S_{\Delta\Lambda} - \frac{1}{3}S_{\Gamma\Gamma}\delta_{\Delta\Lambda}\right),$$

where

(2.14) $$\dot{\lambda} = \gamma\Phi(F)\frac{1}{\sqrt{J_2}}\,.$$

The constant γ is the coefficient of viscosity. The function $\Phi(F)$ should be selected on the basis of the results of experimental investigation into the dynamic properties of the material. In the case of metal plates, the tests show that the function $\Phi(F)$ is particularly useful in the form of the power law

(2.15) $$\Phi(F) = F^{\delta}.$$

The variation of the actual surface of flow during a dynamic process of inealastic strain is determined by the dynamic yield condition

(2.16) $$\Theta = \sqrt{J_2} - k\left[1 + \left(\frac{\sqrt{I_2^p}}{\gamma}\right)^{1/\delta}\right] = 0,$$

where

(2.17) $$I_2^p = \frac{1}{2}\left(\dot{E}^{\,p}_{\Delta\Lambda}\,\dot{E}^{\,p}_{\Delta\Lambda} + \dot{E}^{\,p}_{\Delta\Delta}\,\dot{E}^{\,p}_{\Lambda\Lambda}\right)$$

is the second invariant of the inelastic strain rate tensor. Those rates uniquely determine the stress in the viscoplastic state

$$S_{\Delta\Lambda} = \frac{1}{\dot{\lambda}}\left(\dot{E}^{\,p}_{\Delta\Lambda} + \dot{E}^{\,p}_{\Gamma\Gamma}\delta_{\Delta\Lambda}\right),$$

(2.18)

$$\dot{\lambda} = \frac{\sqrt{I_2^p}}{k\left[1+\left(\frac{\sqrt{I_2^p}}{\gamma}\right)\right]^{1/\delta}}.$$

The active process of viscoplastic strain ceases, if

(2.19) $$F(S_{\Delta\Lambda}) < 0.$$

Then, the process of elastic unloading must be analysed according to Eq. (2.10).

The method used here for formulating equations describing the physical properties of the material is estimated in the paper of DUSZEK [15], in which large deflections of plastic shells are analysed. Transposition of the structure of the law of elastic-viscoplastic flow to the region of finite deformations may be justified from the physical point of view if and only if the real material becomes plastic according to the Huber-Mises-Hencky yield condition expressed in the Kirchhoff space of stresses, that is according to Eq. (2.11). The idea of dynamic evolution of the yield condition expressed by the relation (2.16) and the associated flow law (2.13) will also be correct for such a material. The lack of experimental verification of the above postulates means that the subject of our considerations is a hypothetical material, approximating fairly well the material.

The constitutive relations of elastic-viscoplasticity can be transformed, by passing to the limit for $\gamma \Rightarrow \infty$, into relations of Prandtl-Reuss structure for a perfect elastic-viscoplastic material. Then the scalar factor $\dot{\lambda}$ is determined in a unique manner by the relation $(2.18)_2$. Such a possibility extends in a natural manner the application range of the solutions obtained for dynamic problems of plates.

To estimate the proportion of the participation in the deformation process of states produced by different excitation mechanisms we must assume a definite measure of those states.

Such a measure may be the work performed by the internal forces on the total generalized strains until the moment t_f of reaching by the plate its first maximum deflection amplitude W_f. We shall differentiate, in a natural manner, between states produced by a sheet-membrane mechanism and a flexural mechanism of producing strains. The work of the forces of sheet-membrane state in the time interval assumed is expressed by the relation

$$Q_N = \int_0^{t_f} \int_{\Omega^*} N_{\Delta\Lambda} \dot{L}_{\Delta\Lambda} \, d\Omega^* \, dt. \tag{2.20}$$

For the flexural state we have

$$Q_M = \int_0^{t_f} \int_{\Omega^*} M_{\Delta\Lambda} \dot{K}_{\Delta\Lambda} \, d\Omega^* \, dt. \tag{2.21}$$

The above states of strain constitute, if considered jointly, a shell state in a dynamically deformed plate.

3. Numerical aspects of the analysis

3.1. Discretization of the fundamental equations with respect to time

To solve the set of Eqs. (2.5) of the nonlinear problem of plate dynamics use will be made of an extrapolated difference scheme based on two time layers. It is a conditionally stable scheme enabling us, with a sufficiently small time step Δt, to determine directly the amplitudes $U^\tau_{\Delta n}$, W^τ_n at a moment $t^\tau = \tau \cdot \Delta t$, $\tau = 1, 2, 3, \ldots, T$ on the basis of values known from the preceding instants of time:

$$\begin{aligned} U^\tau_{\Delta n} &= \Delta t^2 \ddot{U}^{\tau-1}_{\Delta n} + 2U^{\tau-1}_{\Delta n} - U^{\tau-2}_{\Delta n}, \\ W^\tau_n &= \Delta t^2 \ddot{W}^{\tau-1}_n + 2W^{\tau-1}_n - W^{\tau-2}_n. \end{aligned} \tag{3.1}$$

The analysis is initiated by using appropriate initiating formulae. For pressure pulses $p(X_\Lambda, t)$ and the assumed initial conditions (2.8) we obtain

$$U^1_{\Delta n} = 0, \qquad W^1_n = \frac{\Delta t^2}{2\mu} P_n(0). \tag{3.2}$$

In the case of an initial velocity pulse, the initiating formulae are

$$U^1_{\Delta n} = 0, \qquad W^1_n = \dot{W}_n(0)\Delta t. \tag{3.3}$$

The quantities $P_n(0)$ and $\dot{W}_n(0)$ should be determined from Eqs. (2.6) and $(2.7)_2$.

An advantage of the method proposed is that it requires a numerical procedure of a recurrence character, owing to which there is no necessity of time-consuming solution of a system of nonlinear algebraic equations for each particular time step. This problem must be tackled in the case of implicit formulations. A fundamental difficulty in applying the explicit method of integration of the equations of motion is the fact that the stability of the method is conditional. The time step Δt ensuring stability must be sufficiently small. It should be observed, however, that the problem under consideration is characterized by a high rate of expansion or vanishing of regions of active plastic flow. A small time step is therefore desirable for correctness of description of the plastic zones evolution.

The determination of $\ddot{U}_{\Delta n}^{\tau-1}$, $\ddot{W}_n^{\tau-1}$ on the basis of Eqs. (2.5) requires evaluation of the integrals involved. In view of the difficulties of determining the physical relations at the level of plate section, some of those integrals must be determined by numerical means. Use will be made of typical procedures of numerical integration based on Gaussian approximate quadratures.

If integration is performed over the region Ω^*, we must establish a network of $I \times J$ nodes, the coordinates of which are $(X_{1i}, X_{2j}) \in \Omega^*$, $i = 1,2,3,\ldots,I$, $j = 1,2,3,\ldots,J$.

Integration over the interval $<-H/2, H/2>$, in order to determine the internal forces from the integral definitions (2.3), requires separation of the K nodes, the coordinates of which are $Z_k \in <-H/2, H/2>$, $k = 1,2,3,\ldots,K$. The numbers I, J, K depend on the order of the quadrature assumed.

It should be observed that the spatial division thus assumed for the region $\Omega = \Omega^* \times <-H/2, H/2>$ plays only a practical role in the procedures of numerical integration and does not discretize the problem in the spatial sense. Moreover, the constant network of nodes i, j, k enables us to study the evaluation of regions of various types of strain.

Denoting the integrand, in a general manner, by $G(X_\Delta, t^\tau)$, the procedure of integration over the rectangular region Ω^* bounded by line segments of lengths A_Δ can be described by the formulae

$$\int_{\Omega^*} G(X_\Delta, t)\, d\Omega^* = \int_0^{A_2} \int_0^{A_1} G(X_1, X_2, t^\tau)\, dX_1\, dX_2$$

$$= \frac{A_1 A_2}{4} \sum_{j=1}^{J} \sum_{i=1}^{I} G(X_{1i}, X_{2j}, t^\tau) \alpha_i \alpha_j, \tag{3.4}$$

$$X_{1i} = A_1(1+\beta_i)/2, \qquad X_2 = A_2(1+\beta_j)/2.$$

The value of the parameters α_i, α_j, β_i, β_j depend on the order of the Gaussian procedure used. The Gaussian approximate quadrature has, as applied to the integral definitions (2.3), the form

$$N_{\Delta\Lambda}(X_{1i},X_{2j},t^\tau) = \frac{H}{2}\sum_{k=1}^{K} S_{\Delta\Lambda}(X_{1i},X_{2j},Z_k,t^\tau)\alpha_k,$$

$$M_{\Delta\Lambda}(X_{1i},X_{2j},t^\tau) = \frac{H}{2}\sum_{k=1}^{K} S_{\Delta\Lambda}(X_{1i},X_{2j},Z_k,t^\tau)Z_k\alpha_k, \tag{3.5}$$

$$Z_k = \frac{H}{2}\beta_k.$$

The work of the forces of the sheet-membrane state and the flexural state will be evaluated by assuming the values of the relevant forces averaged over a time step. On the basis of the relations (2.20) and (2.21) and the procedures (3.4) we obtain

$$Q_N = \frac{A_1A_2}{8}\sum_{\tau=1}^{\tau_f}\sum_{j=1}^{J}\sum_{i=1}^{I}\left(N_{\Delta\Lambda}^{\tau-1} + N_{\Delta\Lambda}^{\tau}\right)\Delta L_{\Delta\Lambda}^{\tau-1,\tau}\alpha_i\alpha_j,$$

$$Q_M = \frac{A_1A_2}{8}\sum_{\tau=1}^{\tau_f}\sum_{j=1}^{J}\sum_{i=1}^{I}\left(M_{\Delta\Lambda}^{\tau-1} + M_{\Delta\Lambda}^{\tau}\right)\Delta K_{\Delta\Lambda}^{\tau-1,\tau}\alpha_i\alpha_j, \tag{3.6}$$

for $\tau_f = t_f/\Delta t$, where

$$\Delta L_{\Delta\Lambda}^{\tau-1,\tau} = L_{\Delta\Lambda}^{\tau} - L_{\Delta\Lambda}^{\tau-1}, \qquad \Delta K_{\Delta\Lambda}^{\tau-1,\tau} = K_{\Delta\Lambda}^{\tau} - K_{\Delta\Lambda}^{\tau-1} \tag{3.7}$$

are the increases in the total generalized deformation.

3.2. The method of determining the stresses

The values of $U_{\Delta n}^{\tau}$ and W_n^{τ} calculated by means of the recurrence formulae (3.1), make it possible to determine the displacement field and its gradients at an instant of time t^τ. To this aim the relations (2.4) are used. The knowledge of the gradients of the displacement field enables us to determine, from the geometrical relations (2.1) and (2.2), the total strain components $E_{\Delta\Lambda}^{\tau}$ and their increments

$$\Delta E_{\Delta\Lambda}^{\tau-1,\tau} = E_{\Delta\Lambda}^{\tau} - E_{\Delta\Lambda}^{\tau-1}. \tag{3.8}$$

Thus, as a result of solving the problem analysed, we know the increase in total strain between the instants of time $t^{\tau-1}$ and t^τ, as well as the state of stress for $t^{\tau-1}$ at all the nodes of the spatial network. Below we shall consider the problem of determining a new state of stress, that is the state at the moment t^τ, on the basis of the values obtained for the increments in strain, Eq. (3.8). The analysis is performed for a typical mode (i, j, k) on the basis of the elastic visco-plastic relations quoted in Sec. 2.

These relations have the form of differential equations and can be solved by appropriate numerical methods. In the difference method the stresses are determined on the basis of explicit or implicit statements. In the elastic-plastic case both methods are discussed in [16]. Some algorithms for determining the stresses by the method of finite elements have been given in [5]. The implicit procedure for determining stresses discussed in [17] will be used in the present paper.

The knowledge of the increases in total strain enables us to determine the stresses

$$S^{\tau}_{\Delta\Lambda} = S^{\tau-1}_{\Delta\Lambda} - \Delta S^{\tau-1,\tau}_{\Delta\Lambda}. \tag{3.9}$$

The increase in stress $\Delta S^{\tau-1,\tau}_{\Delta\Lambda}$ will be calculated on the basis of the incremental form of the relations $(2.10)_2$, by treating the strain increment $\Delta E^{\tau-1,\tau}_{\Delta\Lambda}$ as elastic. The correctness of such procedure is confirmed by the inequality

$$F(S^{\tau}_{\Delta\Lambda}) < 0. \tag{3.10}$$

If this inequality is satisfied, this means that, in the neighbourhood of the node considered, the process is that of elastic strain. It may be a loading or unloading process in regions of pure or secondary elasticity. If the inequality (3.10) is not satisfied, the value of the stresses computed from Eq. (3.9) are treated as trial values $\overline{S}^{\tau}_{\Delta\Lambda}$. It is then necessary to consider the active process of elastic-viscoplastic deformation in an analogous node. To determine the components $S^{\tau}_{\Delta\Lambda}$ satisfying the transient yield condition (2.16) we must decide on the form of separation of the total strain into an elastic and viscoplastic part. This decision depends on the determination (in the stress space) of the direction of the viscoplastic strain increment vector. In the implicit method which is considered, this vector is assumed to be in agreement with the normal to the instantaneous surface of flow at the point determined by the components $S^{\tau}_{\Delta\Lambda}$; the law (2.9) may therefore be rewritten in the form

$$\Delta E^{\tau-1,\tau}_{\Delta\Lambda} = \Delta^e E^{\tau-1,\tau}_{\Delta\Lambda} + \Delta^p E^{\tau-1,\tau}_{\Delta\Lambda}, \tag{3.11}$$

where

$$\Delta^p E^{\tau-1,\tau}_{\Delta\Lambda} = \Delta\lambda^{\tau-1,\tau}\left(S^{\tau}_{\Delta\Lambda} - \frac{1}{3}S^{\tau}_{\Gamma\Gamma}\delta_{\Delta\Lambda}\right). \tag{3.12}$$

The increases in the total strain in Eq. (3.11) are known. The increases in elastic strain determine in a unique manner the increases in stress, in agreement with Eq. $(2.10)_2$. Thus, making use of Eqs. (3.9), (3.11) and (3.12) with the laws $(2.10)_2$ in their incremental form, we obtain the following set of three algebraic equations

$$(3.13)\qquad S^{\tau}_{\Delta\Lambda} = \frac{1}{1+\nu+3E\Delta\lambda^{\tau-1,\tau}}\left[(1+\nu)\overline{S}^{\tau}_{\Delta\Lambda} + \right. \\ \left. + \frac{(1-2\nu)E\Delta\lambda^{\tau-1,\tau}}{1-\nu+E\Delta\lambda^{\tau-1,\tau}}\overline{S}^{\tau}_{\Gamma\Gamma}\delta_{\Delta\Lambda}\right],$$

the unknowns being $S^{\tau}_{\Delta\Lambda}$ and $\Delta\lambda^{\tau-1}, \tau$, $\overline{S}_{\Delta\Lambda}$ denote the known values of the trial stresses. The set of Eqs. (3.13) is made definite by joining the transient yield condition (2.16)

$$(3.14)\qquad \Theta^{\tau}\left(\Delta\lambda^{\tau-1,\tau}\right) = 0.$$

In view of the nonlinear character of the condition (3.14), the quantity $\Delta\lambda^{\tau-1,\tau}$ will be determined by the Newton iteration method.

$$(3.15)\qquad \Delta\lambda^{\tau-1,\tau}_{r+1} = \Delta\lambda^{\tau-1,\tau}_{r} \\ - \left[\frac{\partial\Theta^{\tau}\left(\Delta\lambda^{\tau-1,\tau}_{r}\right)}{\partial\Delta\lambda^{\tau-1,\tau}_{r}}\right]^{-1}\Theta^{\tau}\left(\Delta\lambda^{\tau-1,\tau}_{r}\right) \qquad r = 0,1,2,\ldots .$$

The iteration process is started by assuming an initial value $\Delta\lambda^{\tau-1,\tau} > 0$. The sequence of iterations is broken, if the condition

$$(3.16)\qquad \left|\Delta\lambda^{\tau-1,\tau}_{r+1} - \Delta\lambda^{\tau-1,\tau}_{r}\right| < \varepsilon,$$

is satisfied where ε denotes the assumed accuracy. Numerical solution of the nonlinear equation (3.14) is correctly conditioned, since there exists only one real root $\Delta\lambda^{\tau-1,\tau}$, [17]. Some difficulties may be encountered in establishing the initial value $\Delta\lambda^{\tau-1,\tau}$. The final value of $\Delta\lambda^{\tau-2,\tau-1}$ in the sequence of iterations for the preceding time step is optimum.

3.3. The stability criterion of the method

A fundamental difficulty in establishing an explicit difference scheme (3.1) is connected with the fact of its stability being conditional, and it consists in the necessity of selecting the length of the time step so as to make it shorter than a certain critical time period, which depends on the properties of the system as a whole.

In the process of elastic-viscoplastic deformation considered here the amplitudes of the displacement field remain finite. This means that a permanent tendency to indefinite increase or decrease in the course of the process would be unjustified from the physical point of view. Thus the natural frequencies of the discrete model (3.1) should take only real values. Harmonic analysis of the difference schemes (3.1) leads to a conclusion that this requirement is satisfied, if

$$\Delta t \leq \begin{cases} \Delta t^{\tau}_{\mathrm{cr}\Delta} = \dfrac{2}{\omega_{\Delta n}(t^{\tau})}\,, \\ \Delta t^{\tau}_{\mathrm{cr}Z} = \dfrac{2}{\omega_{\Delta Z}(t^{\tau})}\,. \end{cases} \tag{3.17}$$

The quantities

$$\omega_{\Delta n}(t^{\tau}) = \sqrt{-\frac{\ddot{U}_{\Delta n}}{U_{\Delta n}}}, \qquad \omega_{Zn}(t^{\tau}) = \sqrt{\frac{P_n/\mu - \ddot{W}_n}{W_n}}$$

are frequencies of the n-th mode of nonlinear natural vibration, longitudinal and flexural, respectively. Those frequencies are mutually coupled and depend on the geometrical changes occurring in the plate due to deformation. It follows that the values of the critical time steps (3.17) are variable in time.

By appropriate modification of the difference schemes (3.1) the computation may be carried out with a variable time step. We can also select a constant value of the time step Δt such that the criteria (3.17) may be satisfied during the entire computation process. The latter method is less troublesome and more effective for the problem under consideration. As regards the evaluation of a constant value Δt, solution of an appropriate linear problem, in which the frequencies of the corresponding elastic vibrations can be determined in a direct manner, may be helpful.

4. Discussion of the numerical results

4.1. *Circular plates*

The results of the theoretical analysis made in Sec. 2 may also be applied to a circular plate. To this aim all the fields considered must be transformed to the material system of cylindrical coordinates $\{R, \phi, Z\}$, $(R, \phi) \in \Omega^*$, $Z \in < -H/2, H/2 >$. Because the dimensions of the coordinates $\{R, \phi, Z\}$ are different, the components of the vectors and tensors are replaced with their physical components.

Let us now consider in detail the case of axially symmetric deformation, which is important for engineering problems. The basic functions assumed for the description of the displacement field will be the eigenfunctions of the relevant linear problems. In the case of a displacement field component associated with the direction R, we are concerned with an eigenfunction of elastic vibration of a circular disc, which is expressed by a Bessel function of the first kind and first order, denoted by J_1. The basic function

$$\chi_{Rn}(\rho) = J_1(\beta_n \rho), \qquad \rho = R/A \tag{4.1}$$

satisfies the condition of horizontal immobility of the edge of the plate of radius A, if

$$J_1(\beta_n) = 0. \tag{4.2}$$

An eigenfunction of the problem of flexural elastic vibrations of a circular plate will be used for the description of the deflection

$$\Psi_n(\rho) = J_0(\lambda_n \rho) - \frac{J_0(\lambda_n)}{I_0(\lambda_n)} I_0(\lambda_n \rho). \tag{4.3}$$

The eigenvalues λ_n satisfy the following equations

$$I_1(\lambda_n)J_0(\lambda_n) + I_0(\lambda_n)J_1(\lambda_n) = 0 \tag{4.4}$$

for a clamped plate, and

$$I_1(\lambda_n)J_0(\lambda_n) + I_0(\lambda_n)J_1(\lambda_n) - \frac{2\lambda_n}{1-\nu} I_0(\lambda_n)J_0(\lambda_n) = 0 \tag{4.5}$$

for a plate on hinged supports. The functions I_0, I_1 are modified Bessel functions of the first kind and zero and first order, respectively. The symbol J_0 denotes a Bessel function of the first kind and zero order. Equation (4.5) results from the condition of the radial bending moment becoming zero at the edge of an elastic plate. In the case of elastic-plastic deformation characterized by the formation of regions of active plastic flow, also in the edge zone, the condition (4.5) may remain unsatisfied. This phenomenon is due to a change in value of the Poisson ratio in regions which have become plastic.

In our detailed analysis of the deformation process our attention will be focussed on the interaction between the forces of sheet-membrane and flexural state. The factors to be considered are: the type of edge support, the character of dynamic load, the type of plate material etc.

The object of the present analysis is a circular plate on hinged supports and a plate with clamped edge. In the first case the load will be assumed to be uniform and having the form of a pressure pulse acting over the entire surface, $p(R,t)$ = const. The clamped plate will be loaded by an initial velocity pulse $V_0(R)$ = const generated over the entire surface of the plate or a part of it. The material constants are

$$\begin{aligned} \rho_0 &= 2.7\,\mathrm{t/m^3} && \text{mass density,} \\ E &= 71\,\mathrm{GPa} && \text{Young's modulus,} \\ \sigma_0 &= 290\,\mathrm{MPa} && \text{yield point,} \\ \nu &= 0.3 && \text{Poisson ratio.} \end{aligned}$$

The material of the clamped plate is a type of steel sensitive to strain rate and characterized by the following constants,

$$\begin{aligned} \rho_0 &= 7.8\,\mathrm{t/m^3}, & E &= 210\,\mathrm{GPa}, & \sigma_0 &= 280\,\mathrm{MPa}, \\ \nu &= 0.3, & \gamma &= 350\,\mathrm{s^{-1}}, & \delta &= 5. \end{aligned}$$

The dimensions are the same for both plates,

$$\begin{aligned} A &= 0.10\,\mathrm{m} && \text{plate radius,} \\ H &= 0.006\,\mathrm{m} && \text{plate thickness.} \end{aligned}$$

Numerical analysis has been performed by taking 9 terms of the series (2.4), $N = 9$. Stability of the procedure is ensured if the time step is $\Delta t = 1.0\,\mu\mathrm{s}$. A discussion of the stability and convergence of the method is discussed in greater detail in the reference [9] mentioned above.

Figure 1 shows diagrams of the works Q_N, Q_M performed by the longitudinal forces and the bending moment until the instant t_f, at which its flexural amplitude reaches the first maximum W_f. Those works have been assumed to depend on the load. The other horizontal axis is that of dimensionless values of deflection W_f/H, corresponding to the assumed values of the load. In the case of a plate on hinged supports (Fig. 1a) $p = np_0$ = const, where p_0 is the limit load determined on the basis of a kinematic estimate. As a basis for estimating a deformation range within which sheet-membrane and flexural effects play similar roles we may choose the deflection, for which $Q_N = Q_M$. Let us observe the fact that the differences between particular ranges thus established may be considerable (up to several times the plate thickness), depending on the type of load and the edge support conditions. As regards the plate on hinged supports, the work Q_N is greater than Q_M, beginning from the load $p = p_0$, which corresponds to maximum deflection equal to about H. The membrane state of strain begins to dominate distinctly for deflections of some 3 to 4 times the plate thickness. Beyond this deflection range the work of the bending moments is stabilized at a certain level.

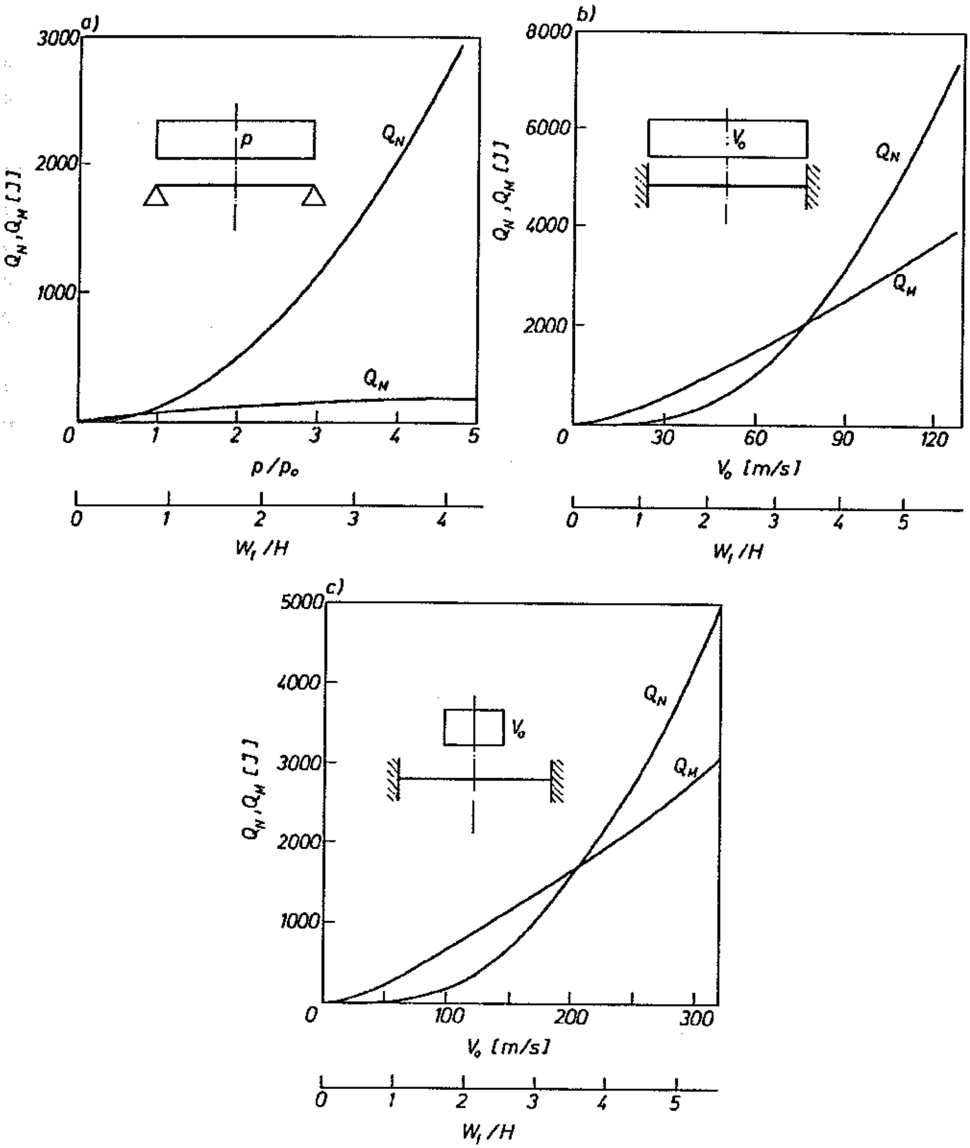


FIG. 1. Dependence of the work of tie-membrane forces Q_N and flexural forces Q_M on the type and intensity of the load and correlated maximum deflection amplitudes of a circular plate.

The interaction between the strain mechanisms under consideration in the course of the process of deformation of a clamped plate subjected to an initial velocity pulse (Fig. 1b, c) is different. In this case a wide range of deflections can be observed ($W_f/H = 3$), in which flexural effects are dominant. Sheet-membrane effects begin to dominate only for much higher deflections. Moreover, the work of the bending moments shows a permanent tendency to increase with increasing pulse V_0. A limitation of the size of the region of action of the load (Fig. 1c) intensifies the flexural character of the dynamic reaction of the plate.

Figure 2 illustrates the variation in time of the internal forces at the midpoint of a plate resting on hinged supports for different values of the load $p/p_0 = 1.0, 4.0$. The values of those forces are referred to the relevant boundary values N_0 and M_0. For a less intense load the character of the variation of the longitudinal forces $N_R = N_\phi$ is for $t \in (0, t_f)$, similar to that of the bending moments $M_R = M_\phi$ (Fig. 2a). The latter reach their extreme values sooner. The moment of occurrence of the first amplitude of the longitudinal forces corresponds to the time t_f. Immediately after the

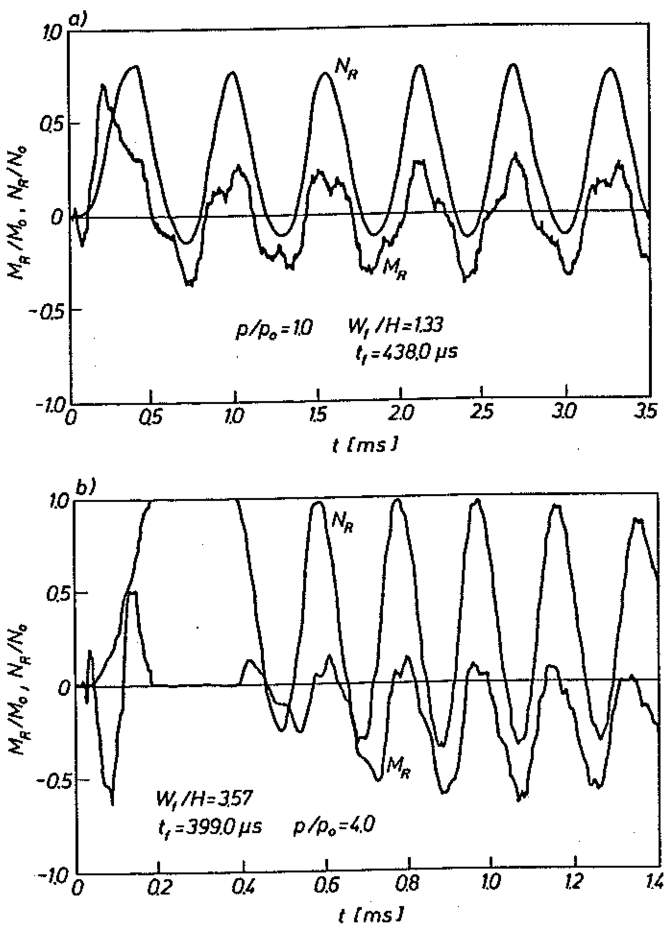
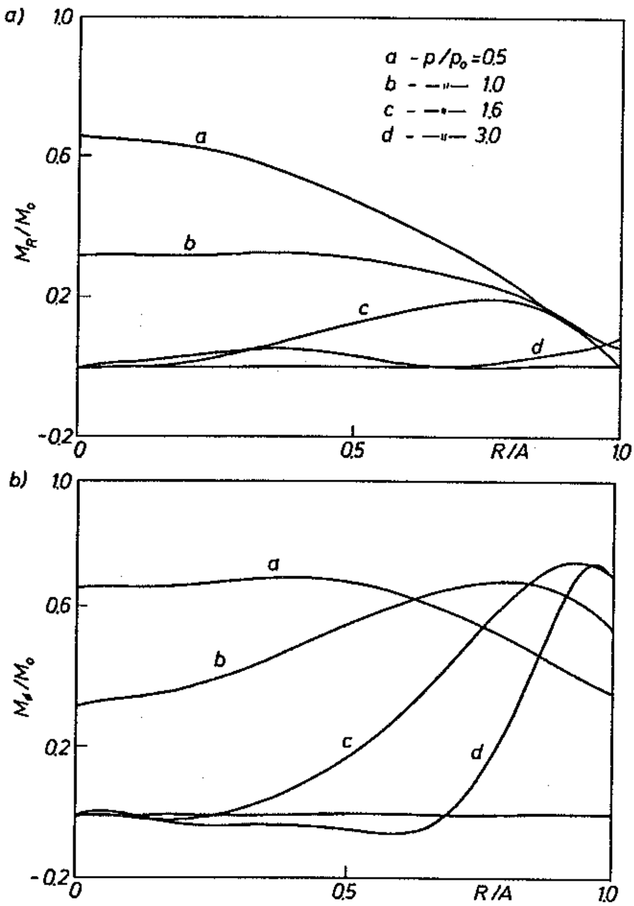
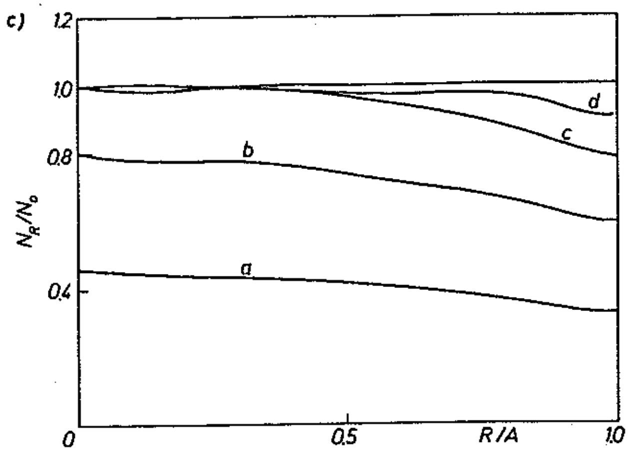


FIG. 2. Time variation of the internal forces at the central point of a circular elastic-plastic plate on a hinged support a) loaded by $p = p_0$; b) loaded by $p = 4p_0$.

deflection of the plate has reached its maximum, the plate undergoes an elastic unloading process. The period of elastic vibration ($t > t_f$) is characterized by symmetric variation of bending moments and by mainly positive longitudinal forces. An increase in intensity of the load up to $p = 4.0p_0$ (Fig. 2b) changes in an essential manner the character of variation of the internal forces in the plate. In the first phase of the motion the loading moments change repeatedly their sign. This is followed by a substantially long period of negative flexural reaction. At the same time the longitudinal forces increase to reach, at the moment $t = 0.5t_f$, a value making the cross-section of the plate. The state of pure membrane reaction (no flexural effects) of in the region about the centre of the plate lasts for about 200 μs. The elastic vibrations are characterized by asymmetric variation of both forces under consideration. Most bending moments are negative and most longitudinal forces – positive. This effect is due to a considerable permanent deformation of the plate.



[Fig. 3]



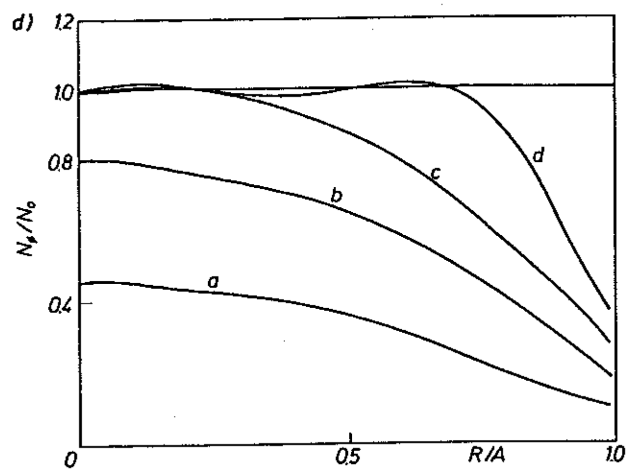


FIG. 3. Distribution of internal forces in a circular plate on a hinged supports at the instant of time coresponding to the first maximum amplitude of deflection, for various intensities of the load $p(R,t) = np_0 = \text{const}$; the case of elastic-plastic material.

The curves in Fig. 3 illustrate the distribution of bending moments in a plate on hinged supports for various load intensities, at the time t_f. With increasing load, a decay of bending moments is observed in the central part of the plate, where the membrane forces reach a value making the cross-section $N_R = N_\phi = N_0$ plastic. A flexural reaction of the plate occurs in the region adjacent to the edge, in interaction with the membrane forces. It does not die out completely, because the plate assumes the form of a shallow shell; the conditions of hinged support are not identical with those of membrane state.

Figure 4 represents the variation, as a function of time, of the internal forces in the central section of a clamped plate loaded over its entire surface by an initial velocity pulse $V_0 = 79.5\,\mathrm{m/s}$. The value of this pulse has been selected so that the maximum deflection of the plate considered might be equal to the maximum deflection of a plate on a hinged support and loaded by a pressure $p = 4p_0$ (Fig. 2b). This enables us to compare the types of variation of the forces considered for deformations of similar values, but produced under different conditions. The initiation of a motion of the plate by a velocity pulse leads to markedly non-stationary development of bending moments (Fig. 4a). The type of generation of longitudinal forces (Fig. 4b) compressing the central region of the plate is interesting. The degree of hardening of the material due to the viscoplastic effects is illustrated by the values of the amplitudes of the forces N_R and M_R, which are much higher

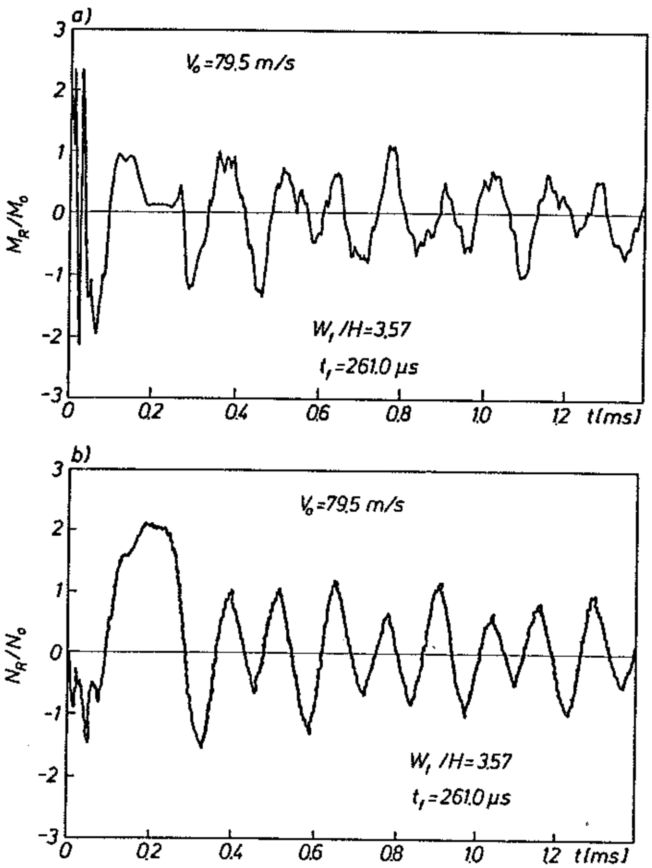


FIG. 4. Time variation of the internal forces at the central point of a clamped circular elastic-viscoplastic plate for an initial velocity pulse $V_0 = 79.5\,\mathrm{m/s}$; a) bending moments, b) normal forces.

than the corresponding values N_0 and M_0. The period of pure membrane-type reaction in the neighbourhood of the centre of the plate is, in this case, much shorter and equal to 71.0 μs. The subsequent vibrations are of the elastic-plastic type, which is proved by the decrease in amplitudes of the longitudinal forces.

4.2. *Rectangular plates*

Let us consider the rectangular plate, the dimensions of which are $2A_1$ and $2A_2$ and thickness H. Let us also assume a uniformly distributed load over the entire plate by an initial velocity pulse V_0. To formulate the problem, the biaxial symmetry of the form of motion will be used. To this aim let us locate the origin of the coordinates $\{x_\alpha, z\}$ at the centre of the undeformed plate. In such a system of coordinates the boundary conditions will be expressed as follows

$$(4.6)\qquad \begin{aligned} &U_\Delta(X_\Lambda, t) = 0, \\ &W(X_\Lambda, t) = W_{,\Delta}(X_\Lambda, t) = 0 \qquad \text{for} \quad X_\Lambda = \pm A_\Lambda. \end{aligned}$$

The displacement field functions are assumed in the form

$$(4.7)\qquad \begin{aligned} U_\Delta(X_\Lambda, t) &= \sum_{m=1}^{M} \sum_{n=1}^{N} U_{mn}(t)\chi_{\Delta mn}(X_\Lambda), \quad \Delta \text{ not summed}, \\ W(X_\Lambda, t) &= \sum_{m=1}^{M} \sum_{n=1}^{N} W_{mn}(t)\Psi_{mn}(X_\Lambda), \end{aligned}$$

where

$$(4.8)\qquad \begin{aligned} \chi_{1mn} &= \sin\frac{m\pi X_1}{A_1}\left(\frac{\operatorname{ch}\beta_n X_2}{\operatorname{ch}\beta_n A_2} - \frac{\cos\beta_n X_2}{\cos\beta_n A_2}\right), \\ \chi_{2mn} &= \left(\frac{\operatorname{ch}\alpha_m X_1}{\operatorname{ch}\alpha_m A_2} - \frac{\cos\alpha_m X_1}{\cos\alpha_m A_1}\right)\sin\frac{n\pi X_2}{A_2}, \\ \Psi_{mn} &= \left(\frac{\operatorname{ch}\alpha_m X_1}{\operatorname{ch}\alpha_m A_1} - \frac{\cos\alpha_m X_1}{\cos\alpha_m A_1}\right)\left(\frac{\operatorname{ch}\beta_n X_2}{\operatorname{ch}\beta_n A_2} - \frac{\cos\beta_n X_2}{\cos\beta_n A_2}\right). \end{aligned}$$

The basic functions (4.8) are combinations of eigenfunctions of the linear problem of longitudinal and flexural vibration of a beam, respectively. Those functions are orthogonal in the region of the plate and satisfy the kinematic boundary conditions (4.6), if

$$(4.9)\qquad \operatorname{th}\alpha_m A_1 + \tan\alpha_m A_1 = 0, \qquad \operatorname{th}\beta_n A_2 + \tan\beta_n A_2 = 0.$$

The set of differential equations of the problem is as follows

(4.10)
$$\ddot{U}_{\Delta mn} = \frac{2}{\mu A_1 A_2} \int_0^{A_2} \int_0^{A_1} N_{\Delta\Lambda} \chi_{\Delta mn,\Lambda} \, dX_1 \, dX_2, \quad \Delta - \text{do not sum up},$$
$$\ddot{W}_{mn} = -\frac{1}{\mu A_1 A_2} \int_0^{A_2} \int_0^{A_1} (N_{\Delta\Lambda} W_{,\Delta} \Psi_{mn,\Lambda} - M_{\Delta\Lambda} \Psi_{mn,\Delta\Lambda}) \, dX_1 \, dX_2 .$$

The initial conditions have the forms

(4.11)
$$U_{\Delta mn}(0) = \dot{U}_{\Delta mn} = W_{mn}(0) = 0.$$
$$\dot{W}_{mn}(0) = \frac{4 \tan \alpha_m A_1 \tan \beta_n A_2}{\alpha_m \beta_n A_1 A_2} V_0 .$$

The results of numerical analysis, which have been obtained for three plates of different dimensions, are collected in Table 1. The material constants of the plate are

$$\rho_0 = 2.7\,\mathrm{t/m^3}, \qquad E = 72\,\mathrm{GPa}, \qquad \sigma_0 = 280\,\mathrm{MPa},$$
$$\nu = 0.3, \qquad \gamma \to \infty.$$

Computation has ben performed for $M \times N = 16$ terms of the series (4.7). With this number of terms the stability of the procedure is ensured, in all the cases of the plate, with a time step $\Delta t = 0.5\,\mu\mathrm{s}$.

Table 1.

No	$2A_1$ [cm]	$2A_2$ [cm]	$\xi = A_2/A_1$	H [cm]
1	12.75	6.36	0.50	0.31
2	12.72	9.57	0.75	0.31
3	12.73	12.72	1.00	0.31

Figure 5 illustrates the dependence of the work of the sheet-membrane forces Q_N, bending moments and torques Q_M on the intensity of the initial velocity field. The character of the interaction between individual strain mechanisms is influenced, in this case, by the initial geometry of the plate which is described by the ratio ξ of its side lengths. The range of deflections in which the effects analysed have a similar share in the deformation process increases with increasing ξ. It may be inferred that an elongated rectangular plate (Fig. 5a) has a stronger tendency to the membrane response than a square plate. Similarly to the case of a clamped circular plate, the values

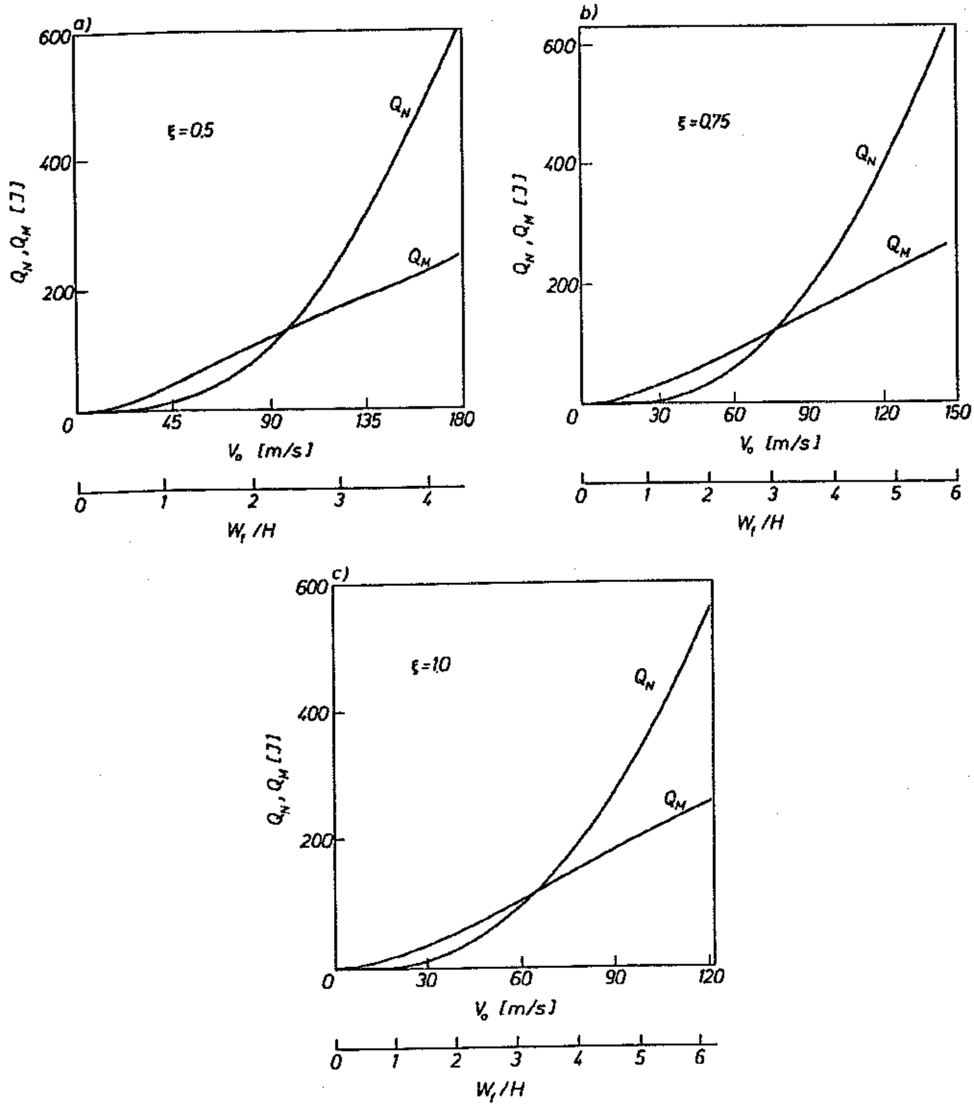


FIG. 5. Dependence of the work of the membrane forces Q_N and the flexural forces Q_M on the intensity of an initial velocity pulse for clamped rectangular elastic-plastic plates; side lengths ratio; a) $\xi = 0.5$, b) $\xi = 0.75$, c) $\xi = 1.0$.

Q_N are higher than Q_M beginning with deflections of order (3 to 4)$\times H$. Figure 6 illustrates the time variation of the inner forces at the centre of a plate, the parameter of which is $\xi = 0.5$. The pulse $V_0 = 146.0\,\mathrm{m/s}$ which is assumed leads to a maximum deflection $W_f/H = 3.56$. The initial phase of motion is characterized by highly non-stationary development of bending moments, in particular the moment M_{22} acting in a direction parallel to the shorter side of the plate (Fig. 6a). Long period of non-flexural reaction of the central region is characteristic for the same direction. This period lasts until the deflection amplitude reaches its maximum value. Later vibrations are of a dissipative character, which is confirmed by the decreasing amplitudes of the longitudinal forces (Fig. 6b).

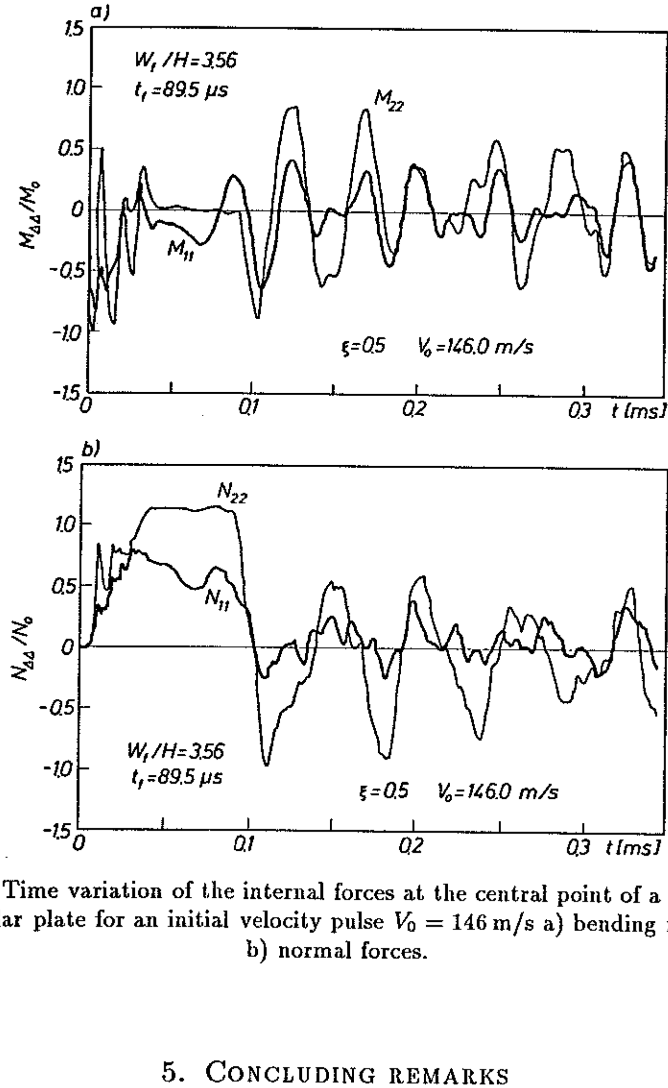


FIG. 6. Time variation of the internal forces at the central point of a clamped rectangular plate for an initial velocity pulse $V_0 = 146$ m/s a) bending moments, b) normal forces.

5. Concluding remarks

The subject of the numerical analysis presented here were circular and rectangular plates, with a thickness of 1/47 to 1/24 of the characteristic in-plane dimension. Such thickness is typical for thin plates. Our attention was focussed on the co-operation between flexural and membrane forces under the conditions of development of active elastic-plastic flow. The range of deflection within which this co-operation is essential depends on the boundary conditions, the type of the load and the form of the plate. It has been found that plates on hinged supports are more sensitive to membrane effects than those with clamped edges. Dynamic loads of long duration produce membrane mechanism of deformation. The motion of a plate forced by an initial velocity pulse is characterized by a highly nonstationary displacement field. In such cases a characteristic feature of the deformation is the flexural mechanism of producing strain. A certain peculiarity of the deformation process of a clamped circular plate loaded by a velocity pulse is the appearance of a region of high compressive forces. This effect is visible in the initial phase of acceleration of the plate and is produced by forms of motion having a negative curvature in the neighbourhood of the central point.

In a deformed, rectangular plate of elongated shape the shell-type reaction is dominated by the tie-arch type response reaction in one direction which is that of the shorter side.

The method of tackling dynamic problems of inelastic plates discussed in the present paper makes possible a more precise description of the phenomena occurring during the process of deformation of such plates. The results obtained show that the assumptions simplifying the analysis should be selected individually for each particular case. Tentative generalizations of such assumptions usually lead to important effects being neglected, which may occur in a particular deformation process.

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