ENGINEERING TRANSACTIONS • *Engng. Trans.* • **43**, 4, 545–552, 1995
Polish Academy of Sciences • Institute of Fundamental Technological Research

FINITE ELEMENT APPLICATIONS TO EVALUATE THE STRESS AND STRAIN FIELD IN THE VICINITY OF AN IMPERFECTION IN THIN SHELLS

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The paper presents the method of calculation of the mechanical and technological parameters of thin spherical shells loaded by internal pressure. The FEM and the plasticity law of the material are introduced to the calculation. The deviations from the designed geometry of spherical shells may be due to fabrication and installation defects, and for that reason it is necessary to evaluate the level of stresses and the nature of the stress redistribution. Numerical results are presented on the diagrams to demonstrate the efficiency of the method and general conclusions are also given at the end of paper.

1. Introduction

Spherical shells often present deviations from the designed geometry with amplitudes of the order of the shell thickness. Such deviations may be due to fabrication and installation defects, accidental loads, damage to the shell. In that case, it may be necessary to evaluate the level of stresses and the nature of the stress redistribution that takes place in the shell with modified geometry.

Typical forms of imperfections can be classified as meridional, circumferential or local, depending on whether curvature errors are introduced in the meridian, in the circumference or both in the meridian and the parallel circle. To evaluate the stress and strain field in the vicinity of an imperfection profile, the usual approach consists in assuming an imperfection profile and carrying out the corresponding deterministic analysis.

The first investigation on the behaviour of imperfect spheres was made by HEYMAN [3]. CALLADINE [4] studied axisymmetric imperfections in cylindrical shells, and the results were extended to spherical shells using Geckeler's assumptions. Fernando and Godoy examined imperfections of shells using the finite element method [5]. The information on the finite element applications to evaluate the stress and strain field in thin shells are reported, for example, in Ref. [2, 6].

2. Nonlinear analysis of imperfect shells of revolution under non-axisymmetric loading

The finite element method is employed to study equilibrium and stability configurations in shells of revolution under internal pressure and non-axisymmetric loading conditions, while the computed load-displacement path may be linear or nonlinear. Bifurcation loads can also be computed from the load-displacement paths. The equations considered in this paper are obtained from those proposed by Sanders [1]. If membrane deformations of the middle surface of the shell ε_{ij} and changes in the curvature of the middle surface of the shell κ_{ij} represent the membrane deformations and changes in the curvature of the middle surface, the strain-displacements relations for the shell of revolution lead to

$$
\begin{aligned}
\varepsilon_{11} &= \frac{\partial u}{\partial \kappa_1} + \frac{w}{r_1} + \frac{1}{2}\left(\beta_1^2 + \beta^2\right),\\
\varepsilon_{22} &= \frac{1}{r}\left(\frac{\partial v}{\partial \kappa_2} - u\sin\phi + w\cos\phi\right) + \frac{1}{2}\left(\beta_2^2 + \beta\right),\\
\varepsilon_{12} &= \frac{1}{2}\left(\frac{\partial v}{\partial \kappa_1} + \frac{1}{r}\left(\frac{\partial u}{\partial \kappa_2} + v\sin\phi\right) + \beta_1\beta_2\right),\\
\kappa_{11} &= -\frac{\partial^2 w}{\partial \kappa_1} + \frac{\partial u}{\partial \kappa_1}\frac{1}{r_1} + u\frac{\partial^2 \phi}{\partial \kappa_1^2},\\
\kappa_{22} &= \frac{1}{r}\left(\frac{1}{r}\left(\frac{\partial^2 w}{\partial \kappa_2^2} - \frac{\partial v}{\partial \kappa_2}\cos\phi\right) + \sin\phi\left(-\frac{\partial w}{\partial \kappa_1} + \frac{u}{r_1}\right)\right),\\
\kappa_{12} &= \frac{1}{2}\left(\frac{1}{rr_1}\frac{\partial u}{\partial \kappa_2} + \left(\sin 2\phi + r\frac{\partial^2 z}{\partial \kappa_2^2}\right)\frac{v}{r^2}\right.\\
&\qquad \left. +2\frac{\cos\phi}{r}\frac{\partial v}{\partial \kappa_1} - \frac{2}{r}\frac{\partial^2 w}{\partial \kappa_1 \partial \kappa_2} - 2\frac{\sin\phi}{r^2}\frac{\partial w}{\partial \kappa_2}\right),
\end{aligned}
\tag{2.1}
$$

where r_1 – radius of the meridian curve, r – parallel radius, u, v, w – displacement in directions of κ_1 (longitudinal coordinate measured along the meridian), κ_2 (angular coordinate along the parallel), κ_3 (longitudinal coordinate along the normal to the shell); ϕ – angular coordinate along the meridian, and

$$
\begin{aligned}
\beta_1 &= -\frac{\partial w}{\partial \kappa_1} + \frac{u}{r_1},\\
\beta_2 &= -\frac{1}{r}\left(\frac{\partial w}{\partial \kappa_2} - v\cos\phi\right),\\
\beta &= \frac{1}{2}\left(\frac{\partial u}{\partial \kappa_1} - v\frac{\sin\phi}{r} - \frac{1}{r}\frac{\partial u}{\partial \kappa_2}\right).
\end{aligned}
\tag{2.2}
$$

Based on the principle of virtual work, the equation of equilibrium can be expressed as:

$$\text{(2.3)}\qquad \frac{1}{2}\iint\limits_S \Big(M_{11}\delta\kappa_{11} + M_{22}\delta\kappa_{22} + 2M_{12}\delta\kappa_{12} + N_{11}\delta\varepsilon_{11} + N_{22}\delta\varepsilon_{22} + 2N_{12}\delta\varepsilon_{12}\Big)ds - \iint\limits_S (p_1\delta u + p_2\delta v + p_3\delta w)\,ds - \int\limits_l r\,(P_1\delta u + P_2\delta v + P_3\delta w + M_1\delta\beta_1)\,dl = 0,$$

where s – area of the middle surface of the shell, l – length of parallel with line loads.

The constitutive relations are the same as those for plane stress, and resultants of forces after integration across the thickness can be expressed as:

$$\text{(2.4)}\qquad \begin{aligned} N_{11} &= C(\varepsilon_{11} + \nu\varepsilon_{22}),\\ N_{22} &= C(\varepsilon_{22} + \nu\varepsilon_{11}),\\ N_{12} &= C(1-\nu)2\varepsilon_{12},\\ M_{11} &= D(\kappa_{11} + \nu\kappa_{22}),\\ M_{22} &= D(\kappa_{22} + \nu\kappa_{11}),\\ M_{12} &= D(1-\nu)2\kappa_{12}, \end{aligned}$$

where C – shell membrane stiffness, D – shell bending stiffness.

The virtual work equation can be written in matrix form

$$\text{(2.5)}\qquad \varphi(a) = \iint\limits_v B^T N\,dv - f = 0.$$

B^T – transpose of a matrix B, where B is defined by the relation

$$\text{(2.6)}\qquad \delta\varepsilon = B\,\delta a$$

and

$$\text{(2.7)}\qquad \begin{aligned} N &= (N_{11}, N_{22}, N_{12}, M_{11}, M_{22}, M_{12}),\\ \varepsilon &= (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \kappa_{11}, \kappa_{22}, \kappa_{12}),\\ a &= (u, v, w). \end{aligned}$$

The operator deformation-displacements matrix B can be decomposed into B_0 independent of nodal displacements a, and nonlinear deformation-displacements matrix B_L dependent on a,

$$\text{(2.8)}\qquad B = B_0 + B_L(a).$$

To solve Eq. (2.5) by a numerical algorithm, it is necessary to know the relation between ψ and a, which is

$$d\psi = \iint_S dB^T N\, ds + \iint_S B^T\, dN ds. \tag{2.9}$$

The differential of N, using Eqs. (2.4) and (2.5), can be written in the form

$$dN = D\, d\varepsilon = D\left(\overline{B}\, da + d\overline{B}a\right) \tag{2.10}$$

and, according to Eq. (2.8),

$$d\overline{B} = dB_L\,. \tag{2.11}$$

Hence, Eq. (2.9) ψ yields

$$d\psi = \iint_S dBN\, ds + \overline{K}\, da, \tag{2.12}$$

where

$$\overline{K} = \iint_S B^T D\left(\overline{B} + B_L\right) ds = K_0 + K_L\,. \tag{2.13}$$

K_0 is the small displacements stiffness matrix

$$K_0 = \iint_S B^T\, D\, B_0\, ds\,, \tag{2.14}$$

while K_L is known as the large displacements matrix and may be written as

$$K_L = \iint_S \left(B_L^T D B_0 + 2B_0^T D B_L + 2B_L^T D\, B_L\right) ds\,. \tag{2.15}$$

The linear and nonlinear deformation components are obtained as follows:

$$\varepsilon = \varepsilon_s + \varepsilon_p = B_0 a + B_L(a)a \tag{2.16}$$

and

$$\delta\varepsilon = \delta\varepsilon_s + \delta\varepsilon_p = B_0\delta a + B_L\delta a\,, \tag{2.17}$$

where

$$\delta a^T = [a_0^u, a_0^v, a_0^w, \dots a_j^u, a_j^v, a_j^w, \dots a_{n-1}^u, a_{n-1}^v, a_{n-1}^w].$$

The solution of Eq. (2.9) in incremental form is derived through a modified Newton-Raphson technique.

The formulation has been extended to deal with an elasto-plastic material and to calculate bifurcation from a nonlinear fundamental path in which the geometric nonlinearity and plasticity are taken into account [7]. The situation is complicated by the fact that different classes of materials exhibit different elasto-plastic characteristics. In this method of calculation the von Mises law, which closely approximates the plastic behaviour of metals, is applied.

The computer program presented in paper [8] has been used to solve the problem considered here.

3. The influence of deviations from the perfect geometry and dimensions

The stress concentrations associated with an imperfection should be considered in the design owing to possible plastic yielding of the material and also in cases in which fatigue is an important design element. Typical forms of imperfections in thin shells of revolution are classified as meridional, circumferential or local imperfections. The usual approach consists in assuming an imperfection profile and performing the corresponding deterministic analysis.

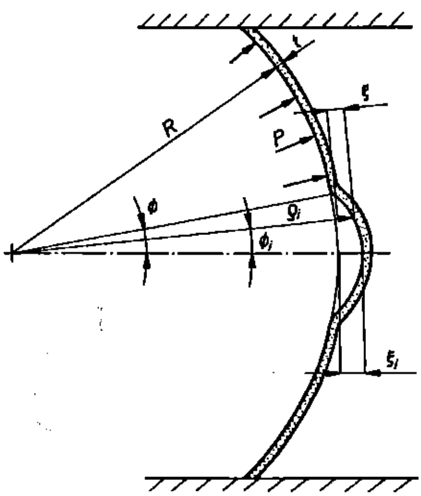

FIG. 1. Spherical shell with local imperfection, loaded by pressure p.

The radial coordinate ϱ_i of a shell with an imperfection of amplitude ξ is given by $\varrho_i = R+\xi$ (Fig. 1). Although the finite element formulation developed is capable of handling any shape of imperfection without difficulties, the studies which follow are restricted to a particular deterministic local imperfection of the form

$$(3.1)\qquad \xi = \frac{\xi_i}{4}\left\{1+\cos\left(\frac{\Pi\phi}{\phi_i}\right)\right\}\left\{1+\cos\left(\frac{\Pi\theta}{\theta_i}\right)\right\},$$

in which ϕ_i is the central angle of imperfection in the meridional direction, θ_i is the angle in the circumferential direction, and ξ_i is the maximum amplitude of imperfection. This particular type of imperfect shell preserves slope continuity at all points, but exhibits discontinuities in curvature at the intersections between perfect and imperfect segments of the shell. Equation (3.1) represents an idealization of real imperfections that may occur in practice. However, the studies reported in the literature on the effects of different profiles of imperfection on the stresses in thin shells of revolution have shown that the stress distributions due to different imperfections do not differ very much from each other. Thus, although the present studies are restricted to one particular imperfection profile, they may be used to foresee the behaviour of the shell with other kinds of imperfections. The numerical analysis has been carried out considering a spherical sector under internal pressure as the basic structure. Twenty elements are used in the computations to model the structure, 10 of which cover the imperfection zone defined by ξ.

The membrane boundary conditions have been applied at the ends of the zone of influence. Studies of convergence indicate that such a discretization represents an adequate approximation to the solution.

An example reported in this section is concerned with a shell having an aspect ratio $R/t = 300$, Young's modulus $E = 2\times 10^5\,\mathrm{N/mm^2}$, Poisson's ratio $\nu = 0.3$ and the parameter of yielding $\sigma_y = 280\,\mathrm{N/mm^2}$. The imperfection considered is defined by the parameters $R/\xi_i = 300$ and $\phi_i = 10°$, which is an imperfection of moderate extent and amplitude equal to the thickness. The membrane stresses have been computed as the equivalent (Huber–Mises) stress at the mid-surface of the shell.

The maximum pressure $p = 400\,\mathrm{N/mm^2}$ has been used to obtain the result shown in Fig. 2. The diagrams illustrate the variation of displacements and membrane stresses with the dimension parameter Φ/Φ_i.

The results show a nonlinear dependence of displacements and stress on the pressure considered, and thus it is necessary to limit the values of pressure to practical values. In this case it is possible to define the influence of the pressure on the stress and displacements of the material.

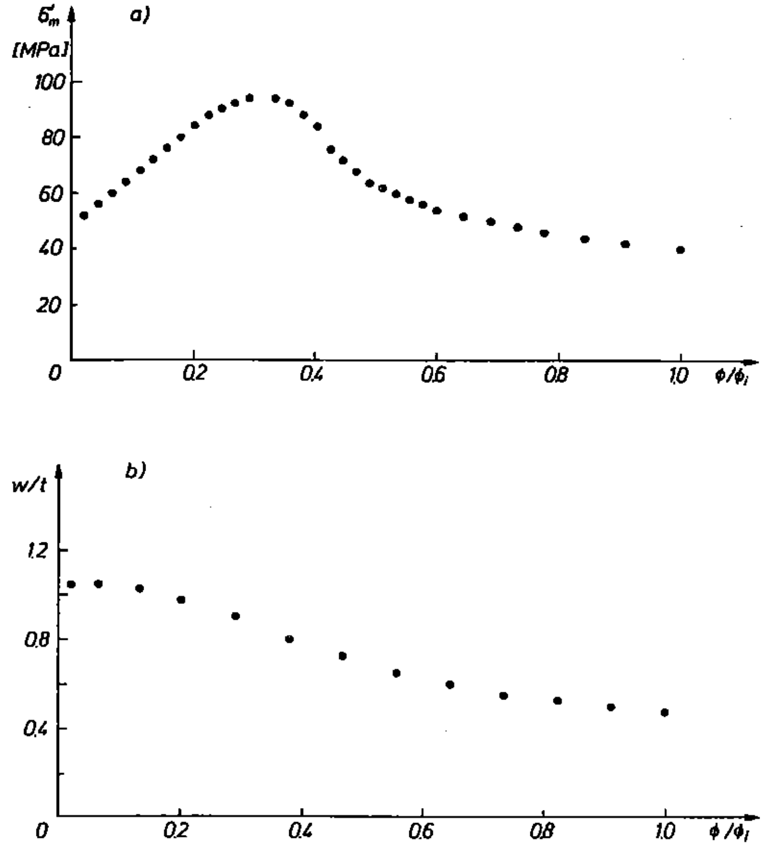

FIG. 2. The membrane stress variation at the mid-surface of the shell (a). The normal displacements variation of the material points of the imperfect shell (b).

4. Conclusions

The finite element method has been applied to the analysis of spherical shells under internal pressure. The example presented in this paper was chosen to illustrate the usefulness of the FEM, and accounting for other kinds of imperfections in the shell is also possible. Thus, for shells with local changes in curvature along the meridian subject to internal pressure, the FEM analysis in which geometric nonlinearity is taken into account should be applied.

In certain cases the finite element method is extremely helpful in verifying the quality of the material of shell under internal pressure. In complex problems, the finite element approach is shown to be the only possible and very useful method.

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Received February 2, 1995.