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VARIATIONAL MODIFICATION OF ŽEMOČKIN CONTACT PROBLEMS SOLUTION METHOD

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The present paper deals with some contact problems. In the well-known Žemočkin method of solving the contact problem of two bodies (structure – elastic half-space), the condition of perfect contact (mutual displacements of two bodies equal zero) is fulfilled only in a chosen number of points. In a modification of the Žemočkin method presented by the authors, the reciprocity of displacements of two bodies is minimized by a functional of potential energy of deformation. Some numerical results concerning the problem of contact of a plate and an elastic half-space based on this solution are included.

1. Introduction

In the well-known Žemočkin collocation method, the contact of two bodies (beam, plate, or a wall with a half-space) is fulfilled in a number of chosen points. Equations expressing this contact have a simple physico-geometric interpretation, though they create a symmetric system of equations only if the following conditions are fulfilled:

(1) The contact domain belonging to the half-space is divided into equal sub-domains (as far as their shapes and sizes are concerned); the contact stress in sub-domains is approximated by a constant value.

(2) The contact stress acting on the plate, beam or wall is replaced by the singular forces that act in specific discrete points.

Solution by the collocation method gives no information about the values of the relative displacements of bodies outside the selected discrete points. Only subsequently, from the results of numerical solution, we can deduce whether it is necessary to refine the division of the contact domain. Generally, we can claim that the collocation method yields no local minimum of the deformation potential energy under the division of contact domain into a finite number of sub-domains. Thus, there must exist such a combination of the planar force impulses approximating the real reaction of the subgrade which, at the same division into the sub-domains, yields better energy balance results.

Thus, we derive the equation system of contact from the conditions of minimum of the expanded modified functional of potential energy of deformation. As it will be seen later, the system will be symmetric even for a non-symmetric division of domains, in which the planar loading impulses act. The diagonal symmetry of the equation system follows from mutuality of the virtual works of the system of planar impulse loads.

2. Assumptions and derivation of basic equations

Let us consider the contact of two Bodies I and II with the surface areas S^{I}, S^{II}. Then let the contact surface be divided into n sub-domains S_r ($r = 1, 2, \ldots n$). $S = S_1 \cup S_2 \cup S_3 \ldots \cup S_r \cup \ldots \cup S_n$. The coordinate Cartesian system will be considered according to the Fig. 1. In each of the sub-domains

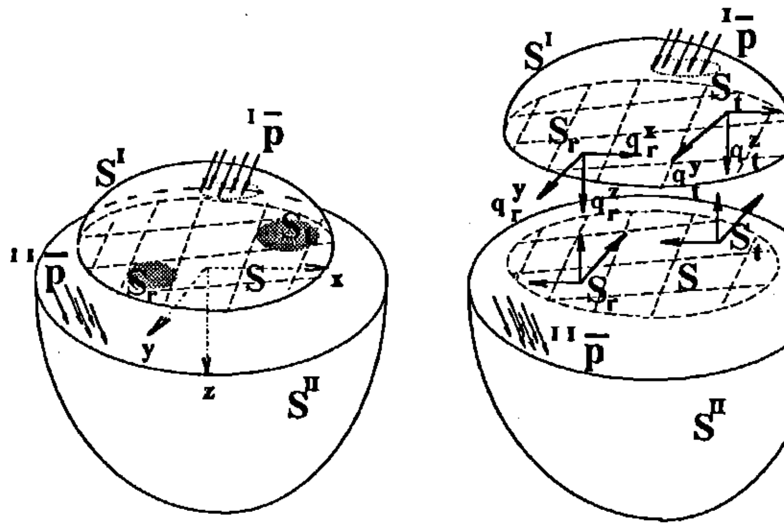


FIG. 1. Contact of two bodies with the contact surface S divided into sub-domains.

S_r we will consider the constant contact stress σ_r^{zj}, simply denoted by q_r^j ($j = x, y, z$); the displacement following from $q_r^j = 1$ in the contact surface S we denote by $u_r^{ij}(x, y, 0)$, where the superscript i ($i = x, y, z$) indicates the corresponding coordinate of the displacement vector; hence it is the r-th base function for the displacement vector in the contact surface S. After the imaginary separation of Bodies I and II, the contact stress q_r^j can be included in the external forces acting on particular bodies. The potential energy of internal forces can be expressed as the work of deformation due to external forces, and it is equal to the sum of works of deformation done by external forces acting on Bodies I and II,

$$W = W^{\mathrm{I}} + W^{\mathrm{II}}. \tag{2.1}$$

Under the assumed coordinate system W^{I} and W^{II} it will be

$$\begin{aligned}
W^{\mathrm{I}} &= \frac{1}{2}\left\{\sum_{r=1}^{n}\left[q_r^i \iint_{S_r}\sum_{t=1}^{n} q_t^j\, {}^{\mathrm{I}}u_t^{ij}(x,y,0)\,dxdy\right]\right. \\
&\qquad \left. + \iint_{S_{\mathrm{I}}} {}^{\mathrm{I}}\overline{p}^i\, {}^{\mathrm{I}}u^i(x,y,z)\,dS\right\} + \sum_{t=1}^{n} q_t^i \iint_{S_t} {}^{\mathrm{I}}u^i(x,y,0)\,{}^{\mathrm{I}}\overline{p}^i\,dxdy, \\
W^{\mathrm{II}} &= -\frac{1}{2}\left\{\sum_{r=1}^{n}\left[q_r^i \iint_{S_r}\sum_{t=1}^{n} q_t^j\, {}^{\mathrm{II}}u_t^{ij}(x,y,0)\,dxdy\right]\right. \\
&\qquad \left. - \iint_{S_{\mathrm{I}}} {}^{\mathrm{II}}\overline{p}^i\, {}^{\mathrm{II}}u^i(x,y,z)\,dS\right\} - \sum_{t=1}^{n} q_t^i \iint_{S_t} {}^{\mathrm{II}}u^i(x,y,0)\,{}^{\mathrm{II}}\overline{p}^i\,dxdy,
\end{aligned} \tag{2.2}$$

where at the contact surfaces S^{I} and S^{II}, displacements ${}^{\mathrm{I}}u_t^{ij}(x,y,0)$ and ${}^{\mathrm{II}}u_t^{ij}(x,y,0)$ in the directions i are produced by load q_t^j.

Let the Body I (after the imaginary separation) have k^{I} (resp. k^{II}) bonds and assume $k^{\mathrm{I}} + k^{\mathrm{II}} \geq 6$.

If $k^{\mathrm{I}} < 6$ (resp. $k^{\mathrm{II}} < 6$), then $6 - k^{\mathrm{I}}$ (resp. $6 - k^{\mathrm{II}}$) bonds must be added to the Body I.

In the civil engineering practice, the most frequent case of the contact problem for the Body I (beam, frame, plate, slab strip, wall) is $k^{\mathrm{I}} = 0$, and the Body II represents the half-space. Displacement of Body I at points with coordinates (x, y, z) can be described by three functions ${}^{\mathrm{I}}\overline{u}^i(x,y,z)$, $i = x, y, z$, for which

$$\begin{Bmatrix} {}^{\mathrm{I}}\overline{u}^x \\ {}^{\mathrm{I}}\overline{u}^y \\ {}^{\mathrm{I}}\overline{u}^z \end{Bmatrix} = \begin{Bmatrix} {}^{\mathrm{I}}u^x \\ {}^{\mathrm{I}}u^y \\ {}^{\mathrm{I}}u^z \end{Bmatrix} + \begin{Bmatrix} {}^{\mathrm{I}}u_0^x \\ {}^{\mathrm{I}}u_0^y \\ {}^{\mathrm{I}}u_0^z \end{Bmatrix}, \tag{2.3}$$

where ${}^{\mathrm{I}}u^i$, $i = x, y, z$ are the displacements of point (x, y, z) of Body I at the primary system, which results from of $6 - k^{\mathrm{I}}$ bonds. ${}^{\mathrm{I}}u_0^i$ are the displacements of points, (x, y, z) of Body I treated as a solid and they are expressed as follows:

$$
(2.4)\qquad \begin{Bmatrix} {}^{\mathrm{I}}u_0^x \\ {}^{\mathrm{I}}u_0^y \\ {}^{\mathrm{I}}u_0^z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & z & -y \\ 0 & 1 & 0 & -z & 0 & x \\ 0 & 0 & 1 & y & -x & 0 \end{bmatrix} \begin{Bmatrix} {}^{\mathrm{I}}u_{00}^x \\ {}^{\mathrm{I}}u_{00}^y \\ {}^{\mathrm{I}}u_{00}^z \\ {}^{\mathrm{I}}\varphi_{00}^x \\ {}^{\mathrm{I}}\varphi_{00}^y \\ {}^{\mathrm{I}}\varphi_{00}^z \end{Bmatrix},
$$

where ${}^{\mathrm{I}}u_{00}^x, {}^{\mathrm{I}}u_{00}^y, \dots, {}^{\mathrm{I}}\varphi_{00}^z$ are the initial parameters in point $(0,0,0)$. Similarly, the displacement of points (x,y,z) of Body II is expressed by the equation

$$
\left\{{}^{\mathrm{II}}\overline{u}\right\} = \left\{{}^{\mathrm{II}}u\right\} + \left\{{}^{\mathrm{II}}u_0\right\}.
$$

Let us extend the quadratic functional W (2.1) by the linear functional $L = L_1 + L_2$, expressing the work of the external load ${}^{\mathrm{I}}\overline{p}$ (resp. ${}^{\mathrm{II}}\overline{p}$) and of the contact stress q_t, in the case of the Body I and II displacement considered as a whole,

$$
(2.5)\qquad \begin{aligned} L_1 &= \iint\limits_{S_{\mathrm{I}}} {}^{\mathrm{I}}u_0^i(x,y,z)\,{}^{\mathrm{I}}\overline{p}^i\,dS + \sum_{t=1}^{n} \iint\limits_{S_t} {}^{\mathrm{I}}u_0^i(x,y,0)\,q_t^i\,dx\,dy, \\ L_2 &= \iint\limits_{S_{\mathrm{II}}} {}^{\mathrm{II}}u_0^i(x,y,z)\,{}^{\mathrm{II}}\overline{p}^i\,dS - \sum_{t=1}^{n} \iint\limits_{S_t} {}^{\mathrm{II}}u_0^i(x,y,0)\,q_t^i\,dx\,dy. \end{aligned}
$$

Let us denote the extended functional $\pi(q_1^i, q_2^i, \dots q_r^i, \dots q_n^i)$ by $W + L$.

The stationarity conditions $(\partial\pi)/(\partial q_r^i) = 0$, $(r = 1, 2, \dots n)$, $(i = x, y, z)$ yield $3n$ conditional equations. Three of them (for x, y, z, respectively) are in the r-th sub-domain in the form:

$$
\begin{aligned}
(2.6)\qquad \frac{\partial\pi}{\partial q_r^i} = {}& \frac{1}{2}\left\{\sum_{t=1}^{n} q_t^j \iint\limits_{S_t} {}^{\mathrm{I}}u_r^{ji}(x,y,0)\,dx\,dy + \sum_{t=1}^{n} q_t^j \iint\limits_{S_r} {}^{\mathrm{I}}u_t^{ij}(x,y,0)\,dx\,dy\right\} \\
& -\frac{1}{2}\left\{\sum_{t=1}^{n} q_t^j \iint\limits_{S_t} {}^{\mathrm{II}}u_r^{ji}(x,y,0)\,dx\,dy + \sum_{t=1}^{n} q_t^j \iint\limits_{S_r} {}^{\mathrm{II}}u_t^{ij}(x,y,0)\,dx\,dy\right\} \\
& + \iint\limits_{S_r} {}^{\mathrm{I}}\overline{p}^i\,{}^{\mathrm{I}}u^i(x,y,0)\,dx\,dy - \iint\limits_{S_r} {}^{\mathrm{II}}\overline{p}^i\,{}^{\mathrm{II}}u^i(x,y,0)\,dx\,dy \\
& + \iint\limits_{S_r} {}^{\mathrm{I}}u_0^i(x,y,0)\,dx\,dy - \iint\limits_{S_r} {}^{\mathrm{II}}u_0^i(x,y,0)\,dx\,dy = 0.
\end{aligned}
$$

Since by the principle of mutuality of virtual works

$$\overline{q}_t^j \iint\limits_{S_t} u_r^{ji}\, dx dy = \overline{q}_r^i \iint\limits_{S_r} u_t^{ij}\, dx dy\,, \tag{2.7}$$

we can transform Eq. (2.6) by expressing $\overline{q}_t^j = \overline{q}_r^i = \overline{1}$ in the following form:

$$\begin{aligned}\sum_{t=1}^{n} q_t^j \iint\limits_{S_r} \left[{}^{\mathrm{I}}u_t^{ij}(x,y,0) - {}^{\mathrm{II}}u_t^{ij}(x,y,0)\right] dx dy \\ + \iint\limits_{S_r} \left[{}^{\mathrm{I}}u_0^{i}(x,y,0) - {}^{\mathrm{II}}u_0^{i}(x,y,0)\right] dx dy \\ = - \iint\limits_{S_r} {}^{\mathrm{I}}\overline{p}^i\, {}^{\mathrm{I}}u^i(x,y,0)\, dx dy + \iint\limits_{S_r} {}^{\mathrm{II}}\overline{p}^i\, {}^{\mathrm{II}}u^i(x,y,0)\, dx dy\,.\end{aligned} \tag{2.8}$$

From $3n$ stationarity conditions, $3n$ equations of the type (2.8) for unknown contact stresses q_t^j ($j = x, y, z$), ($t = 1, 2, 3, \dots n$) follow. Let us consider the case when the Bodies I and II, after the fictitious separation, are not sufficiently reduced in the degrees of freedom. Then, for $12-(k^{\mathrm{I}}+k^{\mathrm{II}})$ unknown parameters of Bodies I and II we define $12-(k^{\mathrm{I}}+k^{\mathrm{II}})$ equilibrium conditions, while $6 \geq 12-(k^{\mathrm{I}}+k^{\mathrm{II}}) \geq 0$. In the equilibrium conditions we consider the zero reactions in additional bonds. For the case when the Body II is a half-space and Body I is the spatial structure having, after the fictitious separation, 6 degrees of freedom, we can write the last six equations:

$$(2.9)\qquad \begin{aligned}
&3n+1) && \sum_{t=1}^{n} \iint\limits_{S_t} q_t^x\, dx dy = -\overline{Q}_x\,,\\
&3n+2) && \sum_{t=1}^{n} \iint\limits_{S_t} q_t^y\, dx dy = -\overline{Q}_y\,,\\
&3n+3) && \sum_{t=1}^{n} \iint\limits_{S_t} q_t^z\, dx dy = -\overline{Q}_z\,,\\
&3n+4) && \sum_{t=1}^{n} \iint\limits_{S_t} q_t^z y\, dx dy = -\overline{M}_x\,,\\
&3n+5) && \sum_{t=1}^{n} \iint\limits_{S_t} q_t^z x\, dx dy = -\overline{M}_y\,,\\
&3n+6) && \sum_{t=1}^{n} \iint\limits_{S_t} (q_t^y x - q_t^x y)\, dx dy = -\overline{M}_z\,.
\end{aligned}$$

The pattern of a symmetric system of Eqs. (2.8) and (2.9) is similar to a system used in the well-known Žemočkin collocation method (the symmetry follows from the theorem of reciprocity of virtual works). The classical Žemočkin method replaces the contact stress acting along the sub-domain S_r by a force X_r [N] acting at the centre of this sub-domain $(x_{0r}, y_{0r}, 0)$ on the surface of the Body I, as well as by an uniform stress $q_r \int_{S_r} dS = X_r$ on the surface of the Body II (halfspace). This system of equations is symmetric (the symmetry follows from the theorem of reciprocity of the displacements) only on the assumption of validity of the cases (1), (2) set up above. The coefficients of equations are determined as follows:

$$\delta_{rt} = {}^{\mathrm{I}}u_t^i(x_{0r}, y_{0r}, 0) + \int_{S_r} q_t \, {}^{\mathrm{II}}u_t^i(x, y, 0)\, dS$$
$$= {}^{\mathrm{I}}u_t^i(x_{0r}, y_{0r}, 0) + {}^{\mathrm{II}}u_t^i(x_{0r}, y_{0r}, 0)\, \Delta x_r \Delta y_r q_t \,.$$

In the collocation method, the points of the contact are usually chosen at the centre of sub-domain. A system of equations obtained by approximating the stress acting at sub-domains S_r is non-symmetric and there is no possibility to make it symmetric. According to the mean value integral theorem, one obtains

$$\iint_{S_r} q_t u_t^i(x, y, 0)\, dx dy = q_t u_t^i(x_\xi, y_\xi, 0) \Delta x_r \Delta y_r \,. \tag{2.10}$$

Note that the coordinates of the point $(x_\xi, y_\xi, 0)$ are not identical with the coordinates of the chosen point of the contact.

3. Example of application

In the following example we will consider the contact of the rectangular plate of dimensions l_x, l_y, h_d with a half-space. At the same time we do not consider the shear stress acting on the contact surface q_r^x, q_r^y. The plate (Body I) will be characterized by physical constant E_d [Pa], (the plate elasticity modulus) ν_d, (Poisson's constants of plate material). The halfspace (Body II) is characterized by constant E_p [Pa], (subgrade modulus of elasticity) and ν_p, (Poisson's constant of subgrade material). The vertical displacement ${}^{\mathrm{I}}u_t^{zz}(x, y, 0) = w_t^d(x, y, 0)$ due to the unit contact stress q_t^z as well as ${}^{\mathrm{I}}u^z(x, y, 0) = w^d(x, y, 0)$ due to the external load ${}^{\mathrm{I}}\overline{p}^z$ are determined by FEM. The sub-domains S_r, $(r = 1, 2, \ldots n)$ are of rectangular

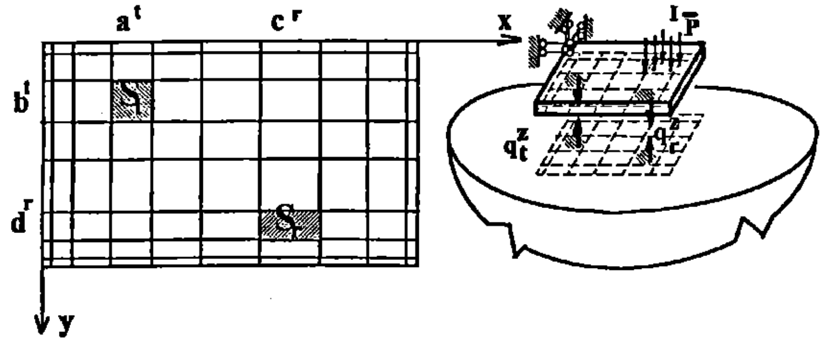


FIG. 2. Division of the rectangular plate into $n_1 \times n_2$ rectangular sub-domains.

shape (Fig. 2) $r = 1, 2, \ldots n$. Let us approximate the plate deflection by the bicubic polynomial

$$w^d(x,y,0) = \langle 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, xy^3 \rangle \{c\}^T. \tag{3.1}$$

The nodal parameters $\left\{ w_i^d, \dfrac{\partial w_i^d}{\partial x}, \dfrac{\partial w_i^d}{\partial y} \right\}$ are found in the well-known manner, while the plate is temporarily supported, what ensures its static and kinematic determinacy. The plate contribution to the t-th term of the r-th equation of the type (2.8) will then be

$$\begin{aligned} &\iint_{S_r} w^d(x,y,0)\,dx\,dy \\ &= \left\langle c^r d^r, \frac{[c^r]^2 d^r}{2}, \frac{c^r [d^r]^2}{2}, \frac{[c^r]^3 d^r}{3}, \frac{[c^r d^r]^2}{4}, \frac{c^r [d^r]^3}{3}, \frac{[c^r]^4 d^r}{4}, \right. \\ &\left. \frac{[c^r]^3 [d^r]^2}{6}, \frac{[c^r]^2 [d^r]^3}{6}, \frac{c^r [d^r]^4}{4}, \frac{[c^r]^4 [d^r]^2}{8}, \frac{[c^r]^2 [d^r]^4}{8} \right\rangle \{C\}^T, \end{aligned} \tag{3.2}$$

where

$$c^r = c_2^r - c_1^r \qquad \text{and} \qquad d^r = d_2^r - d_1^r$$

are the lengths of the r-th part in the direction x, y, respectively, and $\{C\}^T = [S]^{-1}\{\delta\}$. $[S]^{-1}$ is the coordinate matrix with dimensions 12×12.

The vertical displacement of the point of half-space

$$w_t^p(x,y,0) = {}^{\mathrm{II}}u_t^{zz}(x,y,0)$$

produced by load $q_t^{zz} = l$ perpendicular to the half-space plane, acting on the rectangular surface $a^t.b^t$, $(a^t = a_2^t - a_1^t)$, $(b^t = b_2^t - b_1^t)$ (Fig. 3), is obtained

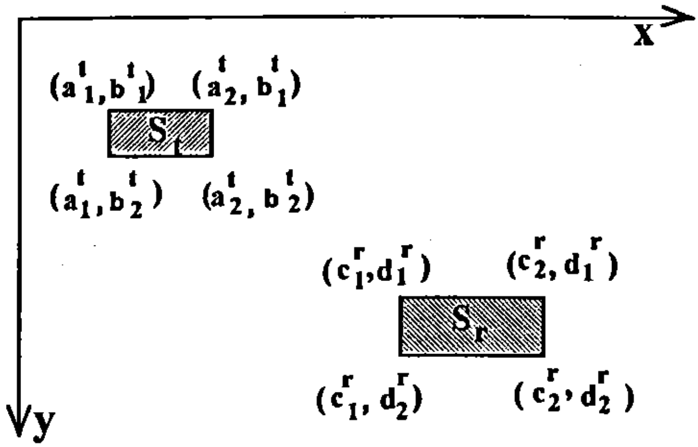


FIG. 3. Dimensions of rectangular domains r and t appearing in (3.3) and (3.4).

e.g. by means of the program MATHEMATICA and is presented in the form

$$(3.3)\qquad w_t^p(x,y,0) = \frac{1-\gamma_p^2}{\pi E_p}\iint\limits_{S_t} \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}}d\xi d\eta$$

$$= \frac{1-\gamma_p^2}{\pi E_p}\sum_{j=1}^{2}\sum_{i=1}^{2}\Big[(y-b_j^t)\ln\left(x-a_i^t+\sqrt{(x-a_i^t)^2+(y-b_j^t)^2}\right)(-1)^{i+j}$$

$$+(x-a_j^t)\ln\left(y-b_i^t+\sqrt{(y-b_i^t)^2+(x-a_j^t)^2}\right)(-1)^{i+j}\Big].$$

After repeated integration of Eq. (3.3) we obtain the half-space contribution to the t-th term of the r-th equation of the type (2.8). We perform this integration by means of the program MATHEMATIKA, and write it in the closed form as follows:

$$(3.4)\qquad \iint\limits_{S_r} w_t^p(x,y,0)\,dxdy$$

$$= \frac{1-\gamma_p^2}{\pi E_p}\Bigg\langle \frac{1}{6}\sum_{k=1}^{2}\sum_{l=1}^{2}\sum_{j=1}^{2}\sum_{i=1}^{2}\Big\{\Big[-2(b_k^t-d_j^r)^2+(a_l^t+c_i^r)^2\Big]$$

$$\cdot\sqrt{(b_k^t-d_j^r)^2+(a_l^t-c_i^r)^2}\Big\}(-1)^{i+j+k+l}$$

$$+\frac{1}{6}\sum_{k=1}^{2}\sum_{l=1}^{2}\sum_{j=1}^{2}\sum_{i=1}^{2}\Big\{\Big[-2(a_k^t-c_j^r)^2+(b_l^t+d_i^r)^2\Big]$$

$$\cdot\sqrt{(a_k^t-c_j^r)^2+(b_l^t-d_i^r)^2}\Big\}(-1)^{i+j+k+l}$$

(3.4)
[cont.]

$$-\frac{1}{2}\sum_{k=1}^{2}\sum_{l=1}^{2}\sum_{j=1}^{2}\sum_{i=1}^{2}\Big[(a_l^t-c_i^r)(b_k^t-d_j^r)^2$$
$$\cdot\ln\Big(c_i^r-a_l^t+\sqrt{(b_k^t-d_j^r)^2+(a_l^t-c_i^r)^2}\,\Big)\Big](-1)^{i+j+k+l}$$
$$-\frac{1}{2}\sum_{k=1}^{2}\sum_{l=1}^{2}\sum_{j=1}^{2}\sum_{i=1}^{2}\Big[(b_l^t-d_i^r)(a_k^t-c_j^r)^2$$
$$\cdot\ln\Big(d_i^r-b_l^t+\sqrt{(a_k^t-c_j^r)^2+(b_l^t-d_i^r)^2}\,\Big)\Big](-1)^{i+j+k+l}\Big\rangle .$$

$(n+1)$, $(n+2)$ and $(n+3)$ term of the r-th equation represents the expression

$$\iint_{S_r}\Big[{}^{\mathrm{I}}u_0^i(x,y,0)-{}^{\mathrm{II}}u_0^j(x,z,0)\Big]\,dx\,dy$$

of the type (2.8).

For the given example we have

$${}^{\mathrm{I}}u_0^x(x,y,0)={}^{\mathrm{I}}u_0^y(x,z,0)={}^{\mathrm{II}}u_0^x(x,y,0)={}^{\mathrm{II}}u_0^y(x,z,0)={}^{\mathrm{II}}u_0^z(x,y,0)=0.$$

The value of ${}^{\mathrm{I}}u_0^z(x,y,0)={}^{\mathrm{I}}u_{00}^z+{}^{\mathrm{I}}\varphi_{00}^x y-{}^{\mathrm{I}}\varphi_{00}^y x$ depends on the method of supporting the plate. The last three terms of the r-th equation (using the notation of equation (3.2)) will be

$$\iint_{S_r}{}^{\mathrm{I}}u_0^z(x,z,0)\,dx\,dy={}^{\mathrm{I}}u_{00}c^r d^r+{}^{\mathrm{I}}\varphi_{00}^x c^r d^r\left(d_1^r+\frac{d^r}{2}\right)-{}^{\mathrm{I}}\varphi_{00}^y c^r d^r\left(c_1^r+\frac{c^r}{2}\right).$$

From 6 equations of the type (2.9) we get 3 equations for the example considered

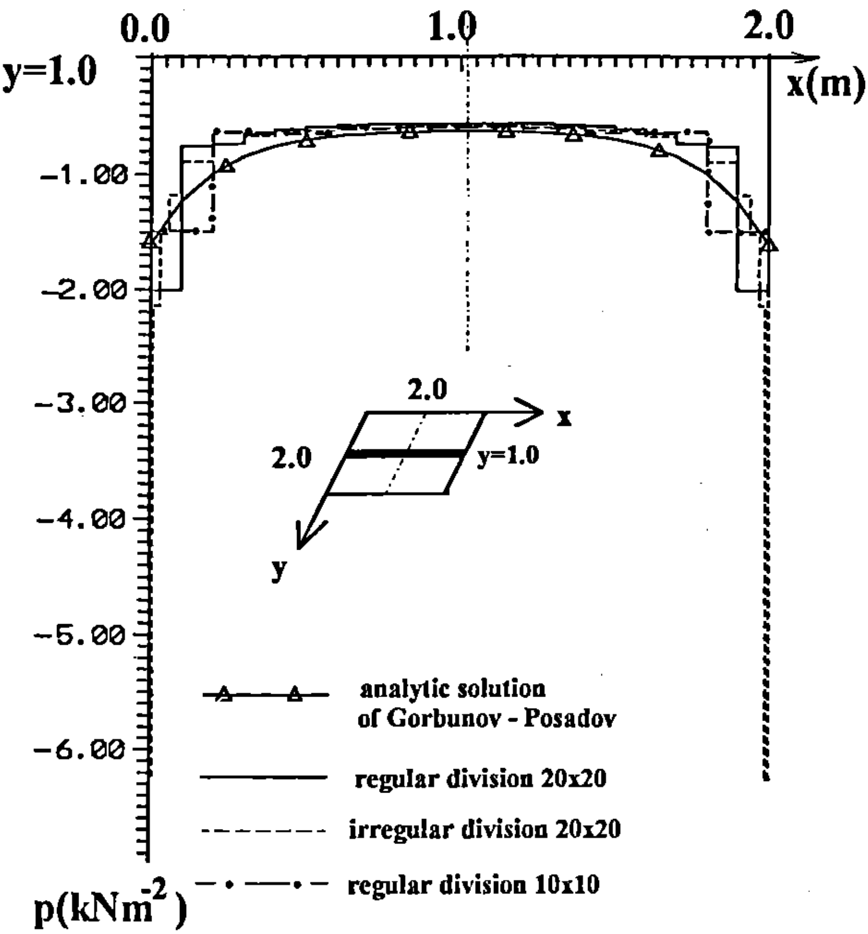
$$n+1)\qquad \sum_{t=1}^{n} q_t^z c^t d^t=-\overline{Q}_z\,,$$
$$n+2)\qquad \sum_{t=1}^{n} q_t^z c^t d^t\left(d_1^t+\frac{d^t}{2}\right)=-\overline{M}_x\,,$$
$$n+3)\qquad \sum_{t=1}^{n} q_t^z c^t d^t\left(c_1^t+\frac{c^t}{2}\right)=-\overline{M}_y\,.$$

According to the presented solution, the computer program has been prepared by the authors of this article. The results were compared with the well-known analytic solution of Gorbunov – Posadov, which is also mentioned in works [1, 2]. Novotný and Hanuska solved the contact problem of the plate and the elastic homogeneous half-space by the collocation Žemočkin method.

They divided the plate into the FEM triangular elements and used the 5th degree polynomial. The contact stress in the triangular domain was approximated by the "pyramid"-like linearly variable function with the peak in the triangle mode.

Graphical illustration of the results (contact stress, deflection, bending moment) of a plate of dimensions $l_x = 2\,\mathrm{m}$, $l_y = 2\,\mathrm{m}$, $h_d = 0.2\,\mathrm{m}$, and with material constants $E_d = 2.65{\cdot}10^7\,\mathrm{kPa}$, $\nu_d = 0.1667$, $E_p = 49{\cdot}10^3\,\mathrm{kPa}$, $\nu_d = 0.4$ is presented in Fig. 4a, 4b, 4c.

The authors of [1] solved the contact problem of the plate assuming various divisions and introducing the presumed boundary values of stress $p(1,1)$, deflection $w(1,1)$ and bending moments $m_x(1,1)$. The solutions approach the presumed values under two various assumptions concerning the approximations.



[FIG. 4a]

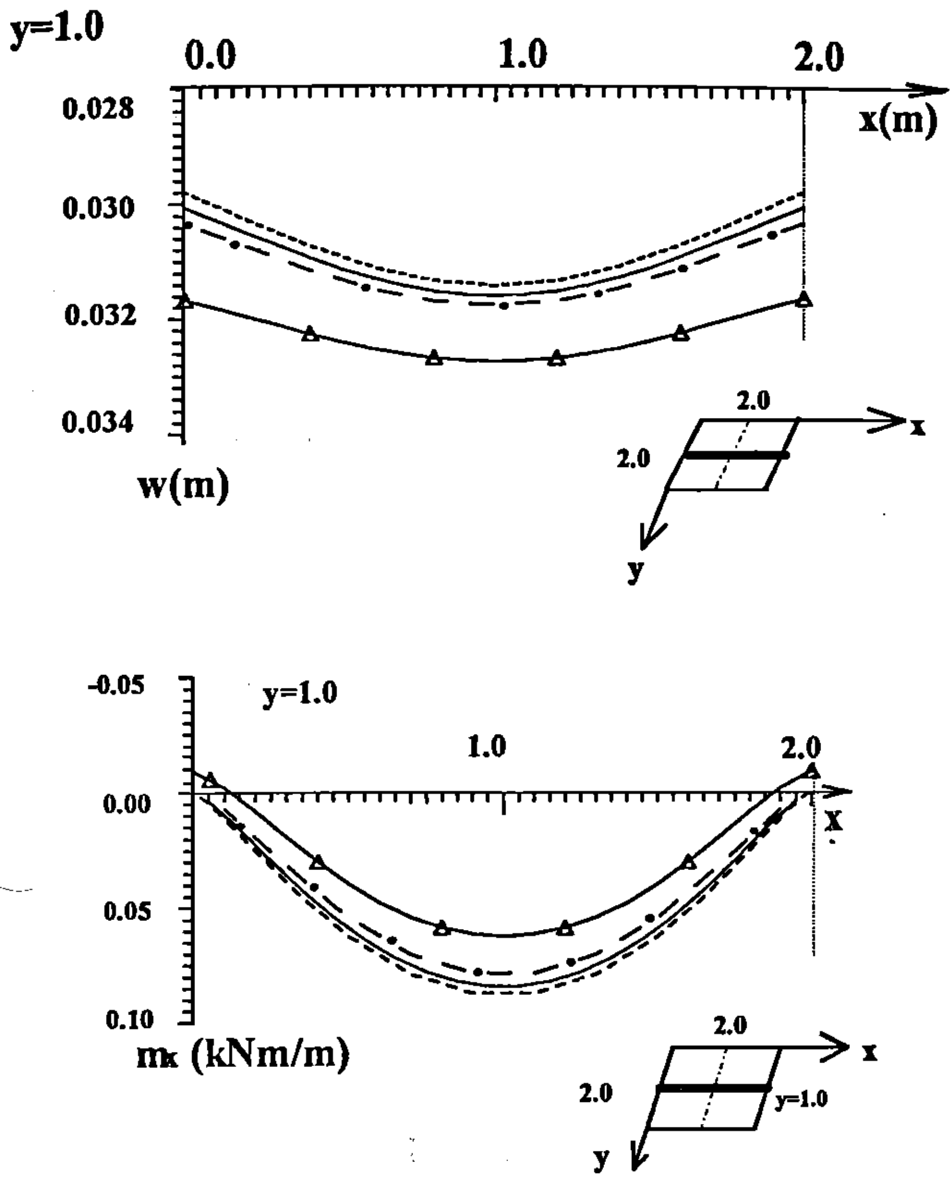


FIG. 4. Comparison of variation of stresses, deflections, and moments m_x in the analytic solution of Gorbunov–Posadov and in the present solution, under various divisions of the contact domain.

Assuming the linear dependence on $n-1$, the boundary results for $\{p(1,1), w(1,1), m_x(1,1)\}$ are $\{-0.5815\,\text{kNm}^{-2}, 0.03155\,\text{m}, 0.0899\,\text{kNm.m}^{-1}\}$. Assuming linear dependence of the error on n, the presumed "more accurate results" for $\{p(1,1), w(1,1), m_x(1,1)\}$ should be $\{-0.5807\,\text{kNm}^{-2}$, $0.03143\,\text{m}$, 0.0914 kNm.m^{-1}}. From the analytic solution of Gorbunov – Posadov, in the centre of the plate the values are $\{p(1,1), w(1,1), m_x(1,1)\} = \{-0.630\,\text{kNm}^{-2}$, 0.0327 m, 0.061 kNm.m^{-1}}. Our solution with three regular divisions of the plate into $n \times n = 10 \times 10$ elements (represented by dotted lines) yields in the centre of the plate the results $\{p(1,1), w(1,1), m_x(1,1)\} = \{-0.60073\,\text{kNm}^{-2}$, 0.03176 m, 0.07819 kNm.m^{-1}}; under the regular division of the plate into $n \times n = 20 \times 20$ elements (represented by the solid line) we obtain $\{p(1,1), w(1,1), m_x(1,1)\} = \{-0.5975\,\text{kNm}^{-2}$, 0.03159 m, 0.08391 kNm.m^{-1}}. With a non-regular division into $n \times n = 20 \times 20$ intervals {0.01, 0.02, 0.03, 0.04, 0.1, 0.1, 0.1, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.1, 0.1, 0.1, 0.04, 0.03, 002, 0.01}, the values are $\{p(1,1), w(1,1), m_x(1,1)\} = \{-0.5956\,\text{kNm}^{-2}$, 0.03142 m, 0.0870 kNm.m^{-1}}.

Comparing the results one can see that the values obtained under the non-regular division 20 × 20 are the closest to the "sharpened" limiting values in [1]. This validates the objections made in [1] concerning the values of bending moments obtained in [4].

In addition to the verification presented, let us present the comparison with the exact solution for a plate subject to non-uniform temperature distribution $\Delta T = T_d - T_h$, where T_d is temperature at $z = h/2$ and T_h at $z = -h/2$. Taking the Kirchhoff – Navier assumptions concerning the plate solution into account, the deflection in the centre of the rectangular plate supported at the corners and loaded by linear temperature distribution ΔT (ignoring the dead weight) is $w^d(0,0) = \dfrac{\alpha_t \Delta T}{2h}\left(\dfrac{l_x^2 + l_y^2}{4}\right)$.

In the following examples the bonds between plate and halfspace are considered as unilateral with respect to pressure. For the plate with dimensions $l_x = l_y = 7.5\,\text{m}$, $h_d = 0.30\,\text{m}$ and with the coefficient of thermal expansion $\alpha_t = 1.2 \cdot 10^{-5}\,{}^\circ\text{C}^{-1}$, for 3 various values of $\Delta T = \{-40°\text{C}, -15°\text{C}, 15°\text{C}\}$, the exact value of deflection in the centre of the plate is $w(0,0) = \{-22.50 \cdot 10^{-3}\,\text{m}, -8.437 \cdot 10^{-3}\,\text{m}, 8.437 \cdot 10^{-3}\text{m}\}$. In our solution we consider the plate with the same geometric and thermal characteristics. The material properties of the plate are $E_d = 2.1 \cdot 10^7\,\text{kPa}$, $\nu_d = 0.24$. For a better comparison of the subgrade effect we will consider three various elasticity moduli of $E_p = \{4.2 \cdot 10^3\,\text{kPa}, 4.2 \cdot 10^4\,\text{kPa}, 4.2 \cdot 10^5\,\text{kPa}\}$, $\nu_p = 0.4$. The division of the rectangular domain of contact into $n \times n = 22 \times 22$ elements is considered as non-regular. The lenghts of rectangular sub-domains are {0.1; 0.2; 0.3; 0.35; 14 × 0.4; 0.35; 0.3; 0.2; 0.1 m}.

Ignoring the dead weight of the plate, the deflection values in the centre of the plate at $\Delta T = \{-40°\text{C}, -15°\text{C}\}$ are $\{-21.90 \cdot 10^{-3}\,\text{m}; -8.2543 \cdot 10^{-3}\text{m}\}$. At $\Delta T = 15°\text{C}$, deflections in the plate corners are $w(3.75; 3.75) =$

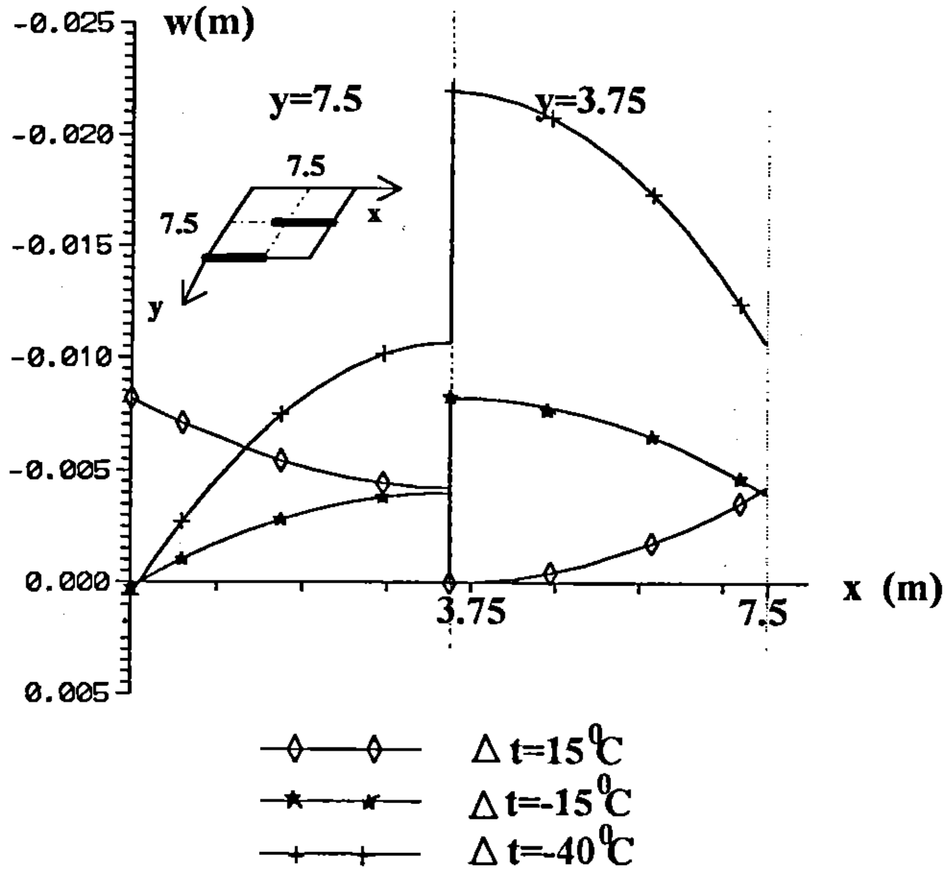


FIG. 5. The plate deflection (ignoring the dead weight) in marked sections under several specified temperature variations.

$-8.2093 \cdot 10^{-3}$ m. The difference of deflections in the centre of the plate and in the corners, according to the exact and approximate solution, are 2.2% and 2.69%, respectively (Fig. 5). The lower value of plate deflection in plate corners at $\Delta T = 15°C$, compared with the deflection in the centre of the plate at $\Delta T = -15°C$, is caused by different sizes of plate elements, as related to the size of contact surface in the centre of the plate and in the plate corners.

The influence of the dead weight and of various elasticity subgrade moduli $E_p = \{4.2 \cdot 10^3; 4.2 \cdot 10^4; 4.2 \cdot 10^5\}$, in case of loading by temperature differences $\Delta T = \{-40°C, -15°C, 15°C\}$, upon the values of contact stresses and deflections in two characteristic sections (symmetry axis and plate edge) is shown in Fig. 6a, b and Fig. 7a, b, c, d, e, f.

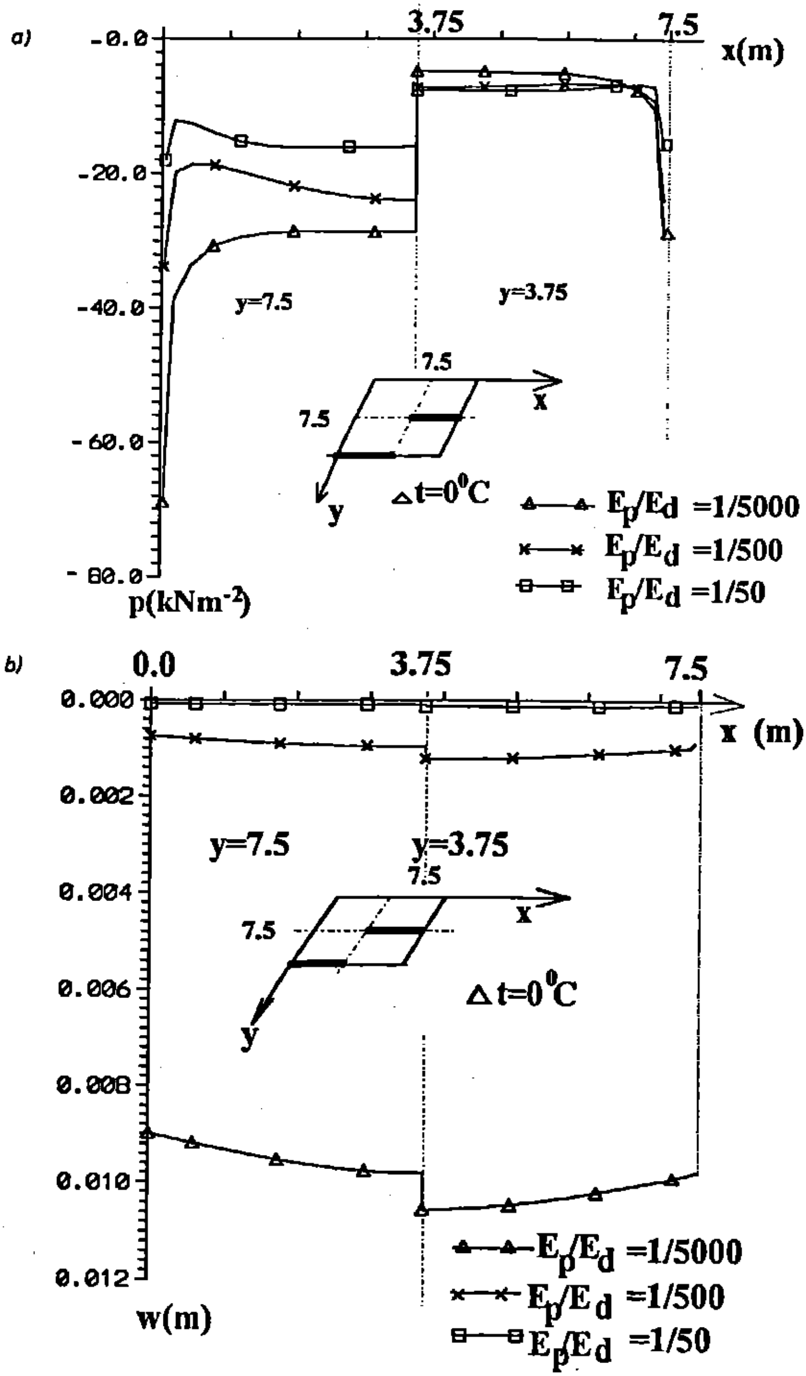
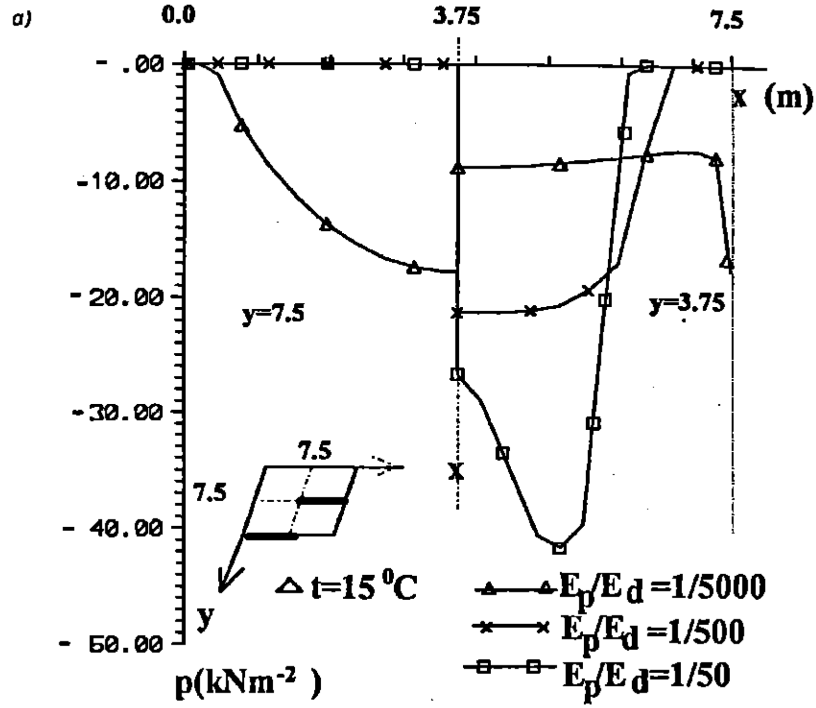
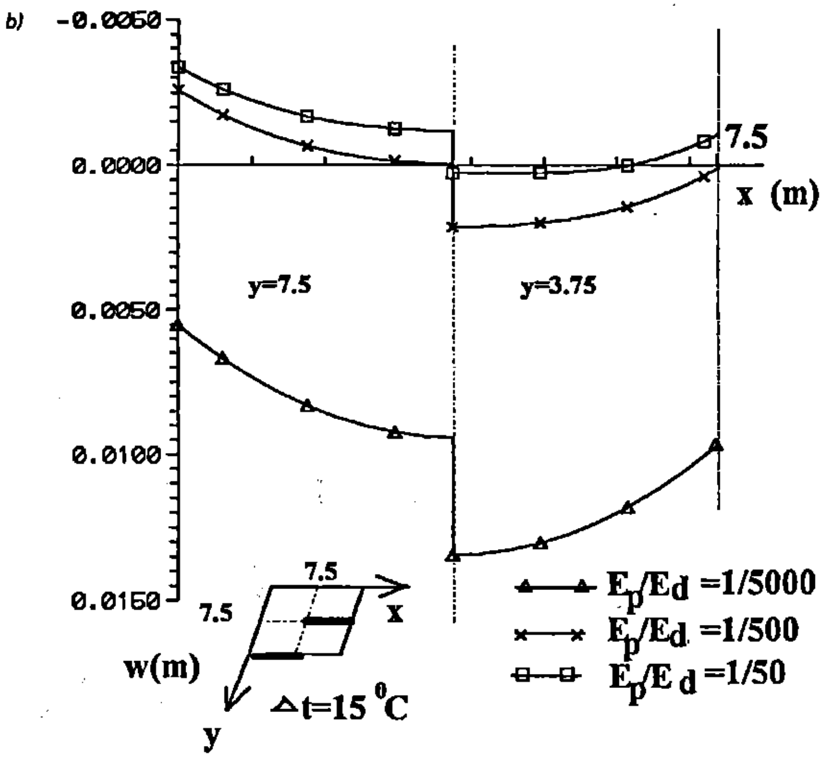
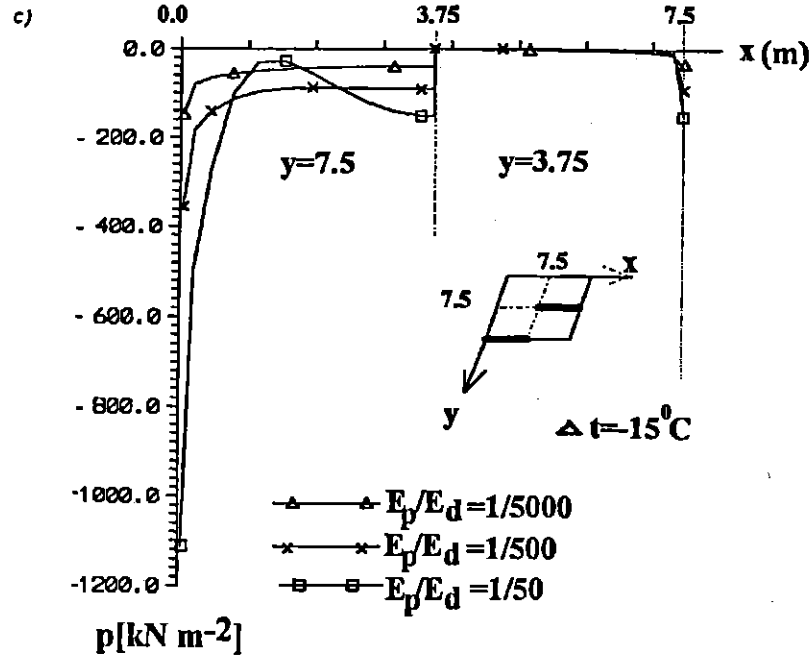


FIG. 6. Contact stress and the plate deflection in marked sections produced by dead weight, at three various elasticity moduli ratios E_d/E_p.

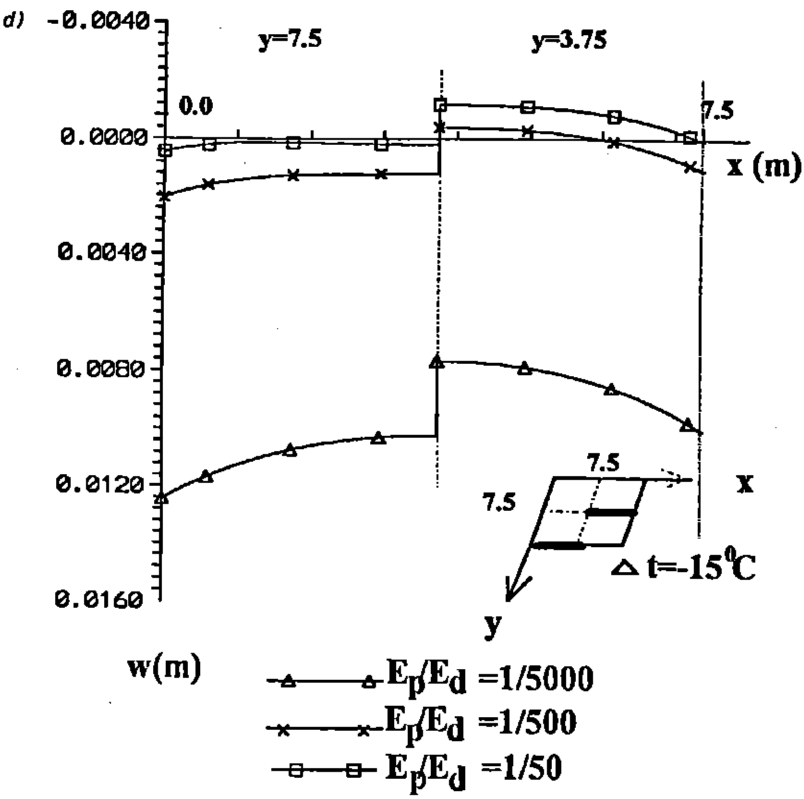

a)
0.0
3.75
7.5
-.00
-10.00
-20.00
-30.00
-40.00
-50.00
x (m)
y=7.5
y=3.75
7.5
7.5
y
p(kNm-2)
△t=15 0C
Ep/Ed =1/5000
Ep/Ed =1/500
Ep/Ed=1/50


b)
-0.0050
0.0000
0.0050
0.0100
0.0150
7.5
x (m)
y=7.5
y=3.75
7.5
7.5
x
w(m)
y
△t=15 0C
Ep/Ed =1/5000
Ep/Ed =1/500
Ep/Ed =1/50

[Fig. 7a, b]



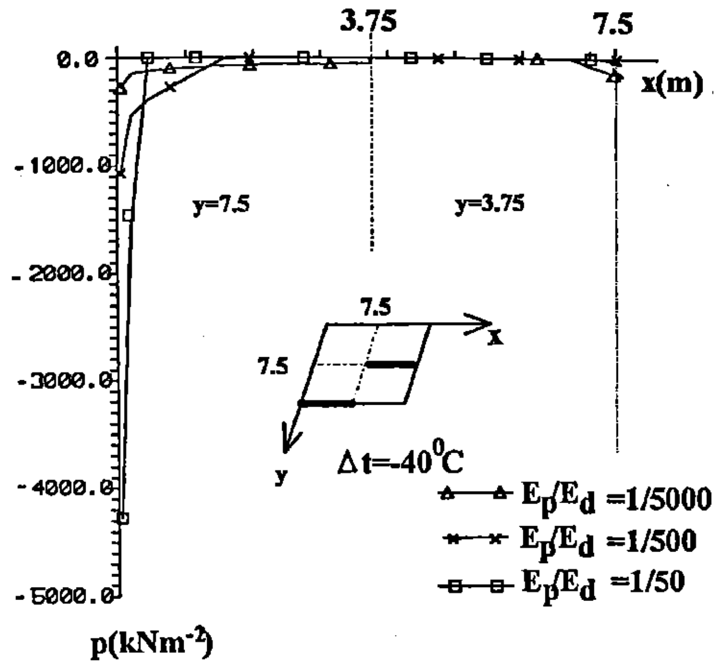
c)
0.0
3.75
7.5
x (m)
0.0
- 200.0
- 400.0
- 600.0
- 800.0
-1000.0
-1200.0
y=7.5
y=3.75
7.5
7.5
x
y
Δ t=-15°C
Ep/Ed =1/5000
Ep/Ed =1/500
Ep/Ed =1/50
p[kN m-2]



d)
-0.0040
y=7.5
y=3.75
0.0
7.5
0.0000
x (m)
0.0040
0.0080
0.0120
0.0160
7.5
7.5
x
Δ t=-15°C
y
w(m)
Ep/Ed =1/5000
Ep/Ed =1/500
Ep/Ed =1/50

[FIG. 7c, d]

e)



f)

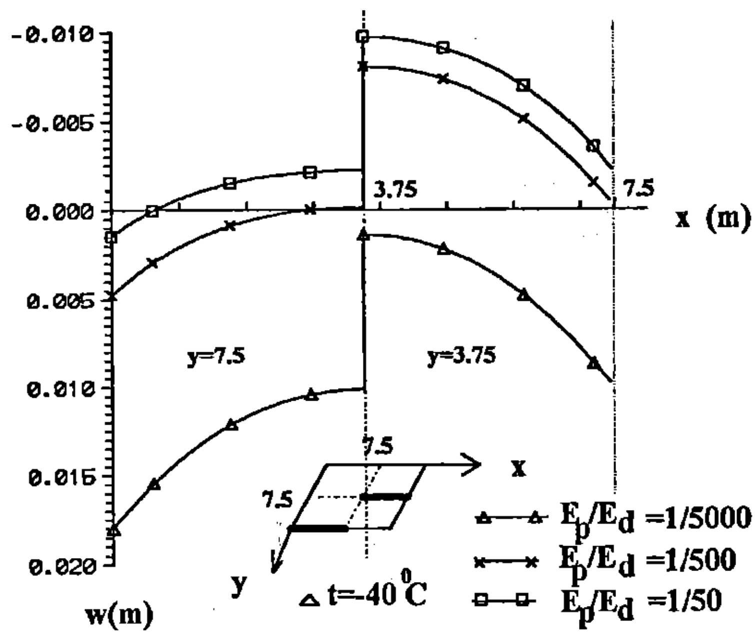


FIG. 7. Contact stress and the plate deflection in marked sections, at three different temperature variations, own weight of the plate and various elasticity moduli E_d/E_p being taken into account.

4. Conclusion

A system of equations derived by the classical Žemočkin collocation method is, in general, not symmetric and there is no possibility to change it to a symmetric system. A variational modification of the Žemočkin method presented by the present authors is derived by means of the theorem of minimum of potential energy of deformation. Unlike the classical Žemočkin method, the system used here is symmetric for any division (shape and size) of the contact domain. This fact leads to a more effective algorithm of solving the system of equations. Using this solution, the autors have prepared a computer program. The numerical results obtained by means of this program are compared with some other results found in the literature [1, 2, 3, 6].

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