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THE PROBLEM OF BENDING OF FIBRE COMPOSITE VISCOELASTIC PLATES

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A method is proposed for solving the problems of bending of prestressed fibre-reinforced viscoelastic plates, subjected to durable loads. The plate considered is made of the primary material (the matrix, phase I), reinforced with densely spaced fibres (phase II) distributed in planes parallel to the middle plane of the plate. The compatibility condition of the two phases is introduced: $\varepsilon = \varepsilon^{\mathrm{I}} = \varepsilon^{\mathrm{II}}$, $\sigma = \sigma^{\mathrm{I}} + \sigma^{\mathrm{II}}$. A numerical algorithm has been prepared, the finite element method being applied. The problem is discretized with respect to time. The paper includes some examples of analysis and inferences.

1. Introduction

The subject of the present work is a theory and an associated computer method for strength analysis of a fibre composite plates made of a viscoelastic material, subject to bending. Under the name of composite plate we shall understand a plate made of a primary filling material, referred to as a matrix, and reinforced with fibres arranged to form one or several families of curves in planes parallel to the middle plane of the plate. The aim of the present work is to elaborate a numerical algorithm for the analysis of stress and strain distribution and its variation as a function of time in a prestressed plate subjected to a prolonged load.

The analysis of the problem of bending of a fibre composite plate is connected with the necessity of overcoming certain difficulties resulting from the complexity of the internal structure of the plate, if prestressed plates are considered in particular, or if the influence of rheological phenomena is studied. The participation of rheological phenomena in the process of bending of a plate is particularly distinct, if synthetic materials are used [1, 2].

In most cases the state of stress and strain in a composite medium is determined on the grounds of continual theories, in which a composite is treated as a homogeneous anisotropic material [3, 4, 5]. The anisotropy constants are determined by homogenization theories [6, 7, 8], which are used, in particular, for the construction of models exhibiting certain regularity. Some new perspectives are opened by the theory of microlocal parameters [9], use being made of the modern apparatus of functional analysis.

In considerations concerning prestressed fibre-reinforced composites, a very important role is played by the description of the interaction between particular phases. In case of a fibre-reinforced composite, which shows no slip between phases, it may be assumed that the strains in the matrix are in agreement with the strains in the fibres in the direction of the fibres [10]. It may also be assumed that the deformations are identical in all the directions [11, 12]. The latter idea has been made use of to derive equations of a plane fibre composite elastic girder [13], and also to solve the problem of a prestressed diphase viscoelastic disc [14, 15].

The present paper is a continuation of earlier considerations concerning composite media, in agreement with the idea presented in [11]. It is assumed that the material of the plate is a diphase composite, one of the phases constituting the matrix (phase I) and the other phase being any number of families of fibres (phase II) distributed sufficiently densely in the fundamental phase to be treated as "fuzzy". The compatibility condition is introduced for those two phases, which means that their deformation is common. From the above it follows that $\varepsilon = \varepsilon^{\mathrm{I}} = \varepsilon^{\mathrm{II}}$, $\sigma = \sigma^{\mathrm{I}} + \sigma^{\mathrm{II}}$.

It is assumed that the load applied to the plate is a prescribed function of time. Discontinuities of the first and second kind are admitted for this function and its derivative with respect to time, which means that the load or the load rate may also vary in time in a stepwise manner, the problem being confined to the quasistatic case, however.

Accepting the necessity of applying numerical methods, the finite element method (FEM) has been selected for spatial discretization [16], and the method described in [17] for discretization in time.

The method presented in [17] consists in subdivision of the time axis into finite intervals of $\vartheta_\tau = t_\tau - t_{\tau-1}$ in length, approximation of the stresses by polynomials, and rigorous solution of the differential equations obtained from the constitutive relation

$$a_0\sigma_\tau + a_1\dot{\sigma}_\tau + a_2\ddot{\sigma}_\tau = b_0\varepsilon_\tau + b_1\dot{\varepsilon}_\tau + b_2\ddot{\varepsilon}_\tau\,, \tag{1.1}$$

where a_τ and b_τ are the viscosity parameters of the material. As a result, we obtain for any instant of time t_τ ($\tau = 0, 1, 2, \ldots$ – denoting the number of that instant) a recurrence equation, in which the stress σ_τ is expressed in terms of ε_τ and depends on the state of the system at the former instant of time $t_{\tau-1}$. The equation expressing the relation between σ_τ and $\dot{\varepsilon}_\tau$ and the state at the moment $t_{\tau-1}$ is formulated separately. In the case of discontinuity of the stress function ($\Delta\sigma_\tau$) or the stress rate ($\Delta\dot{\sigma}_\tau$), it is necessary to evaluate the right-hand limit of those functions: $\sigma'_\tau = \sigma_\tau + \Delta\sigma_\tau$, $\dot{\sigma}'_\tau = \dot{\sigma}_\tau + \Delta\dot{\sigma}_\tau$. If the function expressing the stress and the stress rate is discontinuous, we establish equations relating $\Delta\sigma_\tau$ and $\Delta\dot{\sigma}_\tau$ with $\Delta\varepsilon_\tau$ and $\Delta\dot{\varepsilon}_\tau$.

The relations derived for a one-dimensional state of stress have been generalized to the spatial state of stress [18], then adapted to a diphase viscoelastic disc with distortions [14].

In the present paper, modified constitutive relations will be derived for diphase viscoelastic plates subjected to bending. Those relations are superpositions of the initial and service state due to the external load. Next, a set of equations of the finite elements method will be derived for any instant of time t_τ. Modified finite elements (HCT) according to the GALLAGHER formulation [19], and discussed in detail in [20], will be assumed for discretization in space.

Equations of the finite elements method will be used as a bassis for a program for numerical analysis of fibre composite viscoelastic plates with distortions. The paper includes examples of computation and inferences.

2. Constitutive equations

Figure 1 shows an element of the plate with the r-th family of fibres distributed in a plane at a distance z^r from the middle plane. The number of families of fibres is arbitrary ($r = 1, 2, \ldots$). It is assumed that initial strains $\Delta\overset{\circ}{\varepsilon}{}^r$ can be introduced at $t = 0$ into the fibres of the r-th family. This can be done by prestressing the fibres before the two phases are joined.

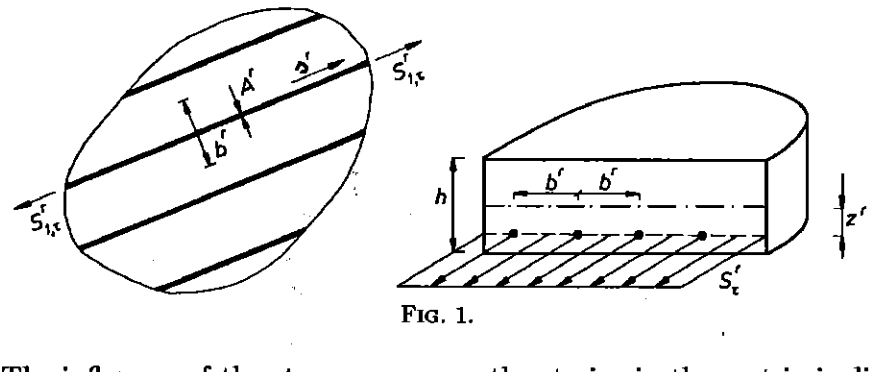

FIG. 1.

The influence of the stresses $\sigma_{33,\tau}$ on the strains in the matrix is disregarded, which leads to the following constitutive relation [14]

$$\boldsymbol{\sigma}_\tau = \mu_\tau \widetilde{\boldsymbol{\lambda}}_\tau \boldsymbol{\varepsilon}_\tau + \mu_\tau \mathbf{G}_\tau \mathbf{e}^*_{\tau-1} + \mathbf{H}_\tau \boldsymbol{\varepsilon}^*_{\tau-1}, \qquad \tau = 1, 2, \ldots, \tag{2.1}$$

the vectors $\boldsymbol{\varepsilon}^*_{\tau-1}$, $\mathbf{e}^*_{\tau-1}$ expressing the state of the system at the preceding instant of time $t_{\tau-1}$

$$\begin{aligned}
&\boldsymbol{\sigma}_\tau = \mathrm{col}(\sigma_{x,\tau}, \sigma_{y,\tau}, \tau_{xy,\tau}), \qquad \boldsymbol{\varepsilon}_\tau = \mathrm{col}(\varepsilon_{x,\tau}, \varepsilon_{y,\tau}, \gamma_{xy,\tau}),\\
&\mathbf{e}^*_{\tau-1} = \mathrm{col}(e_{\tau-1}, \dot{e}_{\tau-1}, s'_{\tau-1}, \dot{s}'_{\tau-1}, \Delta s_{\tau-1}, \Delta\dot{s}_{\tau-1}),\\
&e_{\tau-1} = \varepsilon_{x,\tau-1} + \varepsilon_{y,\tau-1} + \varepsilon_{z,\tau-1}, \qquad s_{\tau-1} = \sigma_{x,\tau-1} + \sigma_{y,\tau-1},\\
&\boldsymbol{\varepsilon}^*_{\tau-1} = \mathrm{col}(\boldsymbol{\varepsilon}^*_{x,\tau-1}, \boldsymbol{\varepsilon}^*_{y,\tau-1}, \boldsymbol{\gamma}^*_{xy,\tau-1}, \boldsymbol{\varepsilon}^*_{z,\tau-1}),\\
&\boldsymbol{\varepsilon}^*_{x,\tau-1} = \mathrm{col}(\varepsilon_{x,\tau-1}, \dot{\varepsilon}_{x,\tau-1}, \sigma'_{x,\tau-1}, \dot{\sigma}'_{x,\tau-1}, \Delta\sigma_{x,\tau-1}, \Delta\dot{\sigma}_{x,\tau-1}),\\
&\mathbf{G}_\tau = \mathrm{col}(\gamma^A_\tau \mathbf{A}^D_\tau - \gamma^D_\tau \mathbf{A}^A_\tau, \gamma^A_\tau \mathbf{A}^D_\tau - \gamma^D_\tau \mathbf{A}^A_\tau, 0),\\
&\mathbf{A}_\tau = [c_{0,\tau}, c_{1,\tau}, a_0 c_{2,\tau} - \gamma_\tau, a_0 c_{3,\tau} + a_1 c_{2,\tau} - \gamma_\tau \vartheta_\tau,\\
&\qquad\qquad a_1 c_{5,\tau} + a_2 c_{6,\tau}, a_2 c_{5,\tau}],\\
&\gamma_\tau = \frac{2}{\vartheta^2_\tau}(a_0 c_{4,\tau} + a_1 c_{3,\tau} + a_2 c_{2,\tau}),
\end{aligned}$$

$$\tilde{\mathbf{X}}_\tau = \begin{bmatrix} \gamma^A_\tau + 2\gamma^D_\tau & \gamma^D_\tau - \gamma^A_\tau & 0\\ \gamma^D_\tau - \gamma^A_\tau & \gamma^A_\tau + 2\gamma^D_\tau & 0\\ 0 & 0 & \gamma^A_\tau + \frac{1}{2}\gamma^D_\tau \end{bmatrix}, \qquad \mu_\tau = \frac{1}{\gamma^D_\tau(2\gamma^A_\tau + \gamma^D_\tau)},$$

$$\mathbf{H}_\tau = \begin{bmatrix} -\frac{1}{\gamma^D_\tau}\mathbf{A}^D_\tau & \mathbf{0} & \mathbf{0} & \mu_\tau(\gamma^D_\tau - \gamma^A_\tau)\mathbf{A}^D_\tau\\ \mathbf{0} & -\frac{1}{\gamma^D_\tau}\mathbf{A}^D_\tau & \mathbf{0} & \mu_\tau(\gamma^D_\tau - \gamma^A_\tau)\mathbf{A}^D_\tau\\ \mathbf{0} & \mathbf{0} & -\frac{1}{\gamma^D_\tau}\mathbf{A}^D_\tau & \mathbf{0} \end{bmatrix}.$$

The upper index D concerns the quantities relating the strain deviator with that of stress, and the upper index A – the spherical parts of the two tensors. The coefficients $c_{i,\tau}$ are listed in [17]. The comma following the index (in the expression $\varepsilon_{x,\tau}$, for instance) does not denote differentiation, its aim being only to separate indices (spatial, for instance) from the ordinal number of the instant of time. The "prime" denotes the right-hand limits at discontinuity points on the axis of time.

Making use of the following relation resulting from the Kirchhoff–Love theory

$$\boldsymbol{\varepsilon}_\tau = z\boldsymbol{\partial} w_\tau + \bar{\boldsymbol{\varepsilon}}_\tau \tag{2.2}$$

we obtain

$$\boldsymbol{\sigma}_\tau = z\mu_\tau \tilde{\mathbf{X}}_\tau \boldsymbol{\partial} w_\tau + \mu_\tau \tilde{\mathbf{X}}_\tau \bar{\boldsymbol{\varepsilon}}_\tau + \boldsymbol{\sigma}^*_{\tau-1}, \tag{2.3}$$

where

$$\boldsymbol{\partial} = \operatorname{col}\left(-\frac{\partial^2}{\partial x^2}, -\frac{\partial^2}{\partial y^2}, -2\frac{\partial^2}{\partial x \partial y}\right),$$

$$\boldsymbol{\sigma}^*_{\tau-1} = \mu_\tau \mathbf{G}_\tau \mathbf{e}^*_{\tau-1} + \mathbf{H}_\tau \boldsymbol{\varepsilon}^*_{\tau-1}.$$

The viscoelastic properties of the fibre phase are described making use of the relations for a uniaxial state of stress. By replacing the forces in single fibres $S^r_{1,\tau}$ with a continuously distributed force we can formulate, for the r-th family, the relation

$$S^r_\tau = \frac{A^r}{b^r \gamma^r_\tau}\left[(\varepsilon^r_\tau - \Delta \overset{\circ}{\varepsilon}{}^r) - \mathbf{A}^r_\tau \boldsymbol{\varepsilon}^{*r}_{\tau-1}\right], \tag{2.4}$$

where

$$\boldsymbol{\varepsilon}^{*r}_{\tau-1} = \operatorname{col}\left(\varepsilon^r_{\tau-1} - \Delta \overset{\circ}{\varepsilon}{}^r, \dot{\varepsilon}^r_{\tau-1}, \sigma'^r_{\tau-1}, \dot{\sigma}'^r_{\tau-1}, \Delta\sigma^r_{\tau-1}, \Delta\dot{\sigma}^r_{\tau-1}\right).$$

We form now a tensor of forces in the fibre phase. Making use of the Kirchhoff–Love theory we express ε^r_τ in terms of w_τ

$$\varepsilon^r_\tau = (\mathbf{T}^r_1)^T(\overline{\boldsymbol{\varepsilon}}_\tau + z^r \boldsymbol{\partial} w_\tau). \tag{2.5}$$

On substituting (2.5) into (2.4) we find

$$\mathbf{S}^r_\tau = \frac{A^r}{b^r \gamma^r_\tau}\mathbf{T}^r_1(\mathbf{T}^r_1)^T(\overline{\boldsymbol{\varepsilon}}_\tau + z^r \boldsymbol{\partial} w_\tau) - \mathbf{S}^{*r}_{\tau-1} - \overset{\circ}{\mathbf{S}}{}^r. \tag{2.6}$$

Here

$$\mathbf{S}^{*r}_{\tau-1} = \frac{A^r}{b^r \gamma^r_\tau}\mathbf{T}^r_1 \mathbf{A}^r_\tau \boldsymbol{\varepsilon}^{*r}_{\tau-1}, \qquad \overset{\circ}{\mathbf{S}}{}^r = \frac{A^r}{b^r \gamma^r_\tau}\mathbf{T}^r_1 \Delta \overset{\circ}{\varepsilon}{}^r,$$

$$\mathbf{T}^r_1 = \operatorname{col}(s^r_x s^r_x,\ s^r_y s^r_y,\ s^r_x s^r_y).$$

The tensors of normal forces and moments acting in the plane of the fibre-reinforced plate are assumed according to the definitions:

$$\mathbf{N}_\tau = \int_{-h/2}^{h/2} \boldsymbol{\sigma}_\tau\, dz + \sum_r \mathbf{S}^r_\tau, \qquad \mathbf{M}_\tau = \int_{-h/2}^{h/2} \boldsymbol{\sigma}_\tau z\, dz + \sum_r \mathbf{S}^r_\tau z^r. \tag{2.7}$$

On substituting the relations (2.3), (2.6) into (2.7) we obtain

$$\begin{aligned}\mathbf{N}_\tau = {} & \left[h\mu_\tau \widetilde{\boldsymbol{\lambda}}_\tau + \sum_r \frac{A^r}{b^r \gamma^r_\tau}\mathbf{T}^r_1(\mathbf{T}^r_1)^T\right]\overline{\boldsymbol{\varepsilon}}_\tau \\ & + \sum_r \frac{A^r}{b^r \gamma^r_\tau} z^r \mathbf{T}^r_1(\mathbf{T}^r_1)^T \boldsymbol{\partial} w_\tau - \mathbf{N}^*_{\tau-1} - \overset{\circ}{\mathbf{N}},\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathbf{M}_\tau = {} & \left[\frac{h^3}{12}\mu_\tau \widetilde{\boldsymbol{\lambda}}_\tau + \sum_r \frac{A^r}{b^r \gamma^r_\tau}(z^r)^2 \mathbf{T}^r_1(\mathbf{T}^r_1)^T\right]\boldsymbol{\partial} w_\tau \\ & + \sum_r \frac{A^r}{b^r \gamma^r_\tau} z^r \mathbf{T}^r_1(\mathbf{T}^r_1)^T \overline{\boldsymbol{\varepsilon}}_\tau - \widetilde{\mathbf{M}}^*_{\tau-1} - \overset{\circ}{\mathbf{M}},\end{aligned} \tag{2.9}$$

where

$$\mathbf{N}^*_{\tau-1} = \sum_r \mathbf{S}^{*r}_{\tau-1} - \int_{-h/2}^{h/2} \boldsymbol{\sigma}^*_{\tau-1}\, dz, \qquad \overset{\circ}{\mathbf{N}} = \sum_r \overset{\circ}{\mathbf{S}}{}^r,$$

$$\widetilde{\mathbf{M}}^*_{\tau-1} = \sum_r \mathbf{S}^{*r}_{\tau-1} z^r - \int_{-h/2}^{h/2} \boldsymbol{\sigma}^*_{\tau-1} z\, dz, \qquad \overset{\circ}{\mathbf{M}} = \sum_r \overset{\circ}{\mathbf{S}}{}^r z^r.$$

The state of bending is the only state to be considered in the present paper. If we reject the stretching state, we have $\mathbf{N}_\tau = \mathbf{0}$, therefore we obtain the following matrix equation, from which we find the stress tensor in the middle plane of the plate [13]

$$\bar{\boldsymbol{\varepsilon}}_\tau - \sum_r z^r \mathbf{C}^r_\tau (\mathbf{T}^r_1)^T \boldsymbol{\partial} w_\tau + \widetilde{\boldsymbol{\varepsilon}}^*_{\tau-1} + \overset{\circ}{\boldsymbol{\varepsilon}}{}^*, \tag{2.10}$$

where

$$\widetilde{\boldsymbol{\varepsilon}}^*_{\tau-1} = \widehat{\mathbf{K}}^{-1}_\tau \mathbf{N}^*_{\tau-1}, \qquad \overset{\circ}{\boldsymbol{\varepsilon}}{}^* = \widehat{\mathbf{K}}^{-1}_\tau \overset{\circ}{\mathbf{N}},$$

$$\mathbf{C}^r_\tau = \mathrm{col}(C^r_{x,\tau}, C^r_{y,\tau}, C^r_{xy,\tau}) = \frac{A^r}{b^r \gamma^r_\tau} \widehat{\mathbf{K}}^{-1}_\tau \mathbf{T}^r_1 .$$

The symbol $\widehat{\mathbf{K}}_\tau = \widehat{\mathbf{K}}^T_\tau = [\widehat{k}_{ij,\tau}]_{3\times 3}$ in Eq. (2.10) denotes a matrix, the elements of which are:

$$\begin{aligned}
\widehat{k}_{11,\tau} &= h\mu_\tau(2\gamma^D_\tau + \gamma^A_\tau) + \sum_r \frac{A^r}{b^r\gamma^r_\tau}(s^r_x)^4,\\
\widehat{k}_{12,\tau} &= h\mu_\tau(\gamma^D_\tau - \gamma^A_\tau) + \sum_r \frac{A^r}{b^r\gamma^r_\tau}(s^r_x)^2(s^r_y)^2,\\
\widehat{k}_{13,\tau} &= \sum_r \frac{A^r}{b^r\gamma^r_\tau}(s^r_x)^3 s^r_y,\\
\widehat{k}_{22,\tau} &= h\mu_\tau(2\gamma^D_\tau + \gamma^A_\tau) + \sum_r \frac{A^r}{b^r\gamma^r_\tau}(s^r_y)^4,\\
\widehat{k}_{23,\tau} &= \sum_r \frac{A^r}{b^r\gamma^r_\tau} s^r_x (s^r_y)^3,\\
\widehat{k}_{33,\tau} &= \frac{1}{2}h\mu_\tau(2\gamma^A_\tau + \gamma^D_\tau) + \sum_r \frac{A^r}{b^r\gamma^r_\tau}(s^r_x)^2(s^r_y)^2.
\end{aligned} \tag{2.11}$$

Substitution of (2.10) into the formula (2.9) leads to the following matrix equation describing the tensor of moments at the instant of time t_τ

$$\mathbf{M}_\tau = \mathbf{D}_\tau \boldsymbol{\partial} w_\tau - \mathbf{M}^*_{\tau-1} - \overset{\circ}{\mathbf{M}}{}^*. \tag{2.12}$$

In the equation (2.12) $\mathbf{D}_\tau$ is the viscoelasticity matrix, $\overset{\circ}{\mathbf{M}}{}^*$ is a tensor of initial state with distortions, and $\mathbf{M}^*_{\tau-1}$ is a tensor describing the state of the system at the preceding moment $t_{\tau-1}$,

$$\mathbf{D}_\tau = \frac{h^3}{12}\mu_\tau\widetilde{\boldsymbol{\lambda}}_\tau + \sum_r \frac{A^r}{b^r\gamma^r_\tau}(z^r)^2\mathbf{T}^r_1(\mathbf{T}^r_1)^T - \sum_r \frac{A^r}{b^r\gamma^r_\tau}z^r\mathbf{T}^r_1(\mathbf{T}^r_1)^T\sum_s z^s\mathbf{C}^s_\tau(\mathbf{T}^s_1)^T,$$

$$\mathbf{M}^*_{\tau-1} = \widetilde{\mathbf{M}}^*_{\tau-1} - \sum_r \frac{A^r}{b^r\gamma^r_\tau}z^r\mathbf{T}^r_1(\mathbf{T}^r_1)^T\widetilde{\boldsymbol{\varepsilon}}^*_{\tau-1},$$

$$\overset{\circ}{\mathbf{M}}{}^* = \overset{\circ}{\mathbf{M}} - \sum_r \frac{A^r}{b^r\gamma^r_\tau}z^r\mathbf{T}^r_1(\mathbf{T}^r_1)^T\,\overset{\circ}{\boldsymbol{\varepsilon}}{}^*.$$

The shear forces are determined by making use of the relation

$$\mathbf{T}_\tau = \boldsymbol{\partial}_1\mathbf{M}_\tau\,, \tag{2.13}$$

where

$$\mathbf{T}_1 = \operatorname{col}(T_{x,\tau}, T_{y,\tau}), \qquad \boldsymbol{\partial}_1 = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}.$$

The strain rate tensor $\dot{\boldsymbol{\varepsilon}}_\tau$ in the matrix and the strain rate tensor in the fibres $\dot{\varepsilon}^r_\tau$ are expressed in terms of the stresses at the instant of time t_τ and by the state of the system at the preceding instant of time $t_{\tau-1}$:

$$\dot{\boldsymbol{\varepsilon}}_\tau = \frac{1}{3}\dot{\boldsymbol{\beta}}_\tau\boldsymbol{\sigma}_\tau + \frac{1}{3}\dot{\gamma}^A_\tau\dot{\mathbf{F}}_\tau\mathbf{e}^*_{\tau-1} + \frac{1}{3\dot{\gamma}^D_\tau}\dot{\mathbf{N}}_\tau\mathbf{e}^*_{\tau-1}, \tag{2.14}$$

$$\dot{\varepsilon}^r_\tau = \dot{\gamma}^r_\tau\sigma^r_\tau + \dot{\mathbf{A}}^r_\tau\boldsymbol{\varepsilon}^{*r}_{\tau-1}\,. \tag{2.15}$$

In the above equations: $\dot{\boldsymbol{\beta}}_\tau$, $\dot{\mathbf{N}}_\tau$, $\dot{\mathbf{F}}_\tau$, $\dot{\mathbf{A}}_\tau$, are matrices of viscoelasticity parameters and result from the formulae given in [17] and [14].

The strain component $\varepsilon_{z,\tau}$ is expressed in terms of the components $\varepsilon_{x,\tau}$ and $\varepsilon_{y,\tau}$ and the state of the system at the preceding instant of time $t_{\tau-1}$. The strain rate $\dot{\varepsilon}_{z,\tau}$ is determined in a similar manner.

The formula for the strain rate tensor results directly from the approximate formula for the stress function within the time interval ϑ_τ [17]. For the matrix and the fibres we have:

$$\dot{\boldsymbol{\sigma}}_\tau = \frac{2}{v_\tau}(\boldsymbol{\sigma}_\tau - \boldsymbol{\sigma}'_{\tau-1}) - \dot{\boldsymbol{\sigma}}'_{\tau-1}\,, \tag{2.16}$$

$$\dot{\sigma}^r_\tau = \frac{2}{v_\tau}(\sigma^r_\tau - \sigma'^r_{\tau-1}) - \dot{\sigma}'^r_{\tau-1}\,. \tag{2.17}$$

The formulae derived above enable us to calculate the left-hand limits of the stress and stress rate functions in the matrix and the fibres. If at the moment t_τ ($\tau = 0, 1, 2, \ldots$) there is discontinuity of those functions, the right-hand limits of the stresses and stress rates must be evaluated for the matrix and the fibres:

$$\tag{2.18} \begin{aligned} \boldsymbol{\sigma}'_\tau &= \boldsymbol{\sigma}_\tau + \Delta\boldsymbol{\sigma}_\tau\,, & \dot{\boldsymbol{\sigma}}'_\tau &= \dot{\boldsymbol{\sigma}}_\tau + \Delta\dot{\boldsymbol{\sigma}}_\tau\,,\\ \sigma'^r_\tau &= \sigma^r_\tau + \Delta\sigma^r_\tau\,, & \dot{\sigma}'^r_\tau &= \dot{\sigma}^r_\tau + \Delta\dot{\sigma}^r_\tau\,. \end{aligned}$$

Stepwise increments of stress and strain and their rates for a plane state of stress are related by the equations [14]:

$$\tag{2.19} \Delta\boldsymbol{\sigma}_\tau = \mu_0 \widetilde{\boldsymbol{\lambda}}_0 \Delta\boldsymbol{\varepsilon}_\tau\,,$$

$$\tag{2.20} \Delta\dot{\boldsymbol{\sigma}}_\tau = \mu_0 \widetilde{\boldsymbol{\lambda}}_0 \Delta\dot{\boldsymbol{\varepsilon}}_\tau + \mu_0^2 \widetilde{\boldsymbol{\eta}}_0 \Delta\boldsymbol{\varepsilon}_\tau\,,$$

where

$$\begin{aligned} \Delta\boldsymbol{\sigma}_\tau &= \operatorname{col}(\Delta\sigma_{x,\tau}\,, \Delta\sigma_{y,\tau}\,, \Delta\tau_{xy,\tau}),\\ \Delta\boldsymbol{\varepsilon}_\tau &= \operatorname{col}(\Delta\varepsilon_{x,\tau}\,, \Delta\varepsilon_{y,\tau}\,, \Delta\gamma_{xy,\tau}), \end{aligned}$$

$$\boldsymbol{\eta}_0 = \begin{bmatrix} -2\gamma_0^A\dot{\gamma}_0^D(\gamma_0^A+\gamma_0^D) - \gamma_0^D\gamma_0^D(3\dot{\gamma}_0^A + 2\dot{\gamma}_0^D) & 2\gamma_0^A\dot{\gamma}_0^D(\gamma_0^A+\gamma_0^D) - \gamma_0^D\gamma_0^D(3\dot{\gamma}_0^A + \dot{\gamma}_0^D & 0\\ 2\gamma_0^A\dot{\gamma}_0^D(\gamma_0^A+\gamma_0^D) - \gamma_0^D\gamma_0^D(3\dot{\gamma}_0^A + \dot{\gamma}_0^D) & -2\gamma_0^A\dot{\gamma}_0^D(\gamma_0^A+\gamma_0^D) - \gamma_0^D\gamma_0^D(3\dot{\gamma}_0^A + 2\dot{\gamma}_0^D) & 0\\ 0 & 0 & -0.5\dot{\gamma}_0^D(2\gamma_0^A+\gamma_0^D)^2 \end{bmatrix}.$$

The corresponding relations for the fibres are:

$$\tag{2.21} \begin{aligned} \Delta S_0^r &= \frac{A^r}{b^r\gamma_0^r}(\Delta\varepsilon_0^r - \Delta\overset{\circ}{\varepsilon}{}^r),\\ \Delta\dot{S}_0^r &= \frac{A^r}{b^r\gamma_0^r}\Delta\dot{\varepsilon}_0^r - \frac{A^r\dot{\gamma}_0^r}{b^r(\gamma_0^r)^2}(\Delta\varepsilon_0^r - \Delta\overset{\circ}{\varepsilon}{}^r), \qquad \tau = 0, \end{aligned}$$

$$\tag{2.22} \begin{aligned} \Delta S_\tau^r &= \frac{A^r}{b^r\gamma_0^r}\Delta\varepsilon_\tau^r\,,\\ \Delta\dot{S}_\tau^r &= \frac{A^r}{b^r\gamma_0^r}\Delta\dot{\varepsilon}_\tau^r - \frac{A^r\dot{\gamma}_0^r}{b^r(\gamma_0^r)^2}\Delta\varepsilon_\tau^r\,, \qquad \tau = 1, 2, \ldots\,. \end{aligned}$$

By assuming the increases in forces and force rates in the plane of the fibres to be continuous, and proceeding as was done above, we derive the formulae enabling us to calculate the increment of the force tensor and its rate in the fibre phase.

If the plate is subjected to bending only, the increment of the normal forces $\Delta \mathbf{N}_\tau$ in the plane of the plate and the increment of the normal force rate $\Delta \dot{\mathbf{N}}_\tau$ become also zero:

$$\Delta \mathbf{N}_\tau = \int_{-h/2}^{h/2} \Delta \boldsymbol{\sigma}_\tau \, dz + \sum_r \Delta \mathbf{S}_\tau^r = \mathbf{0}, \tag{2.23}$$

$$\Delta \dot{\mathbf{N}}_\tau = \int_{-h/2}^{h/2} \Delta \dot{\boldsymbol{\sigma}}_\tau \, dz + \sum_r \Delta \dot{\mathbf{S}}_\tau^r = \mathbf{0}.$$

These formulae are used to derive the following equations determining the increments of strain tensors and their rates in the middle plane of the plate

$$\Delta \bar{\boldsymbol{\varepsilon}}_\tau = -\sum_r z^r \Delta \mathbf{C}^r (\mathbf{T}_1^r)^T \boldsymbol{\partial} \Delta w_\tau + \Delta \mathring{\boldsymbol{\varepsilon}}^*, \tag{2.24}$$

$$\Delta \dot{\bar{\boldsymbol{\varepsilon}}}_\tau = -\sum_r z^r \Delta \mathbf{C}^r (\mathbf{T}_1^r)^T \boldsymbol{\partial} \Delta \dot{w}_\tau + \Delta \dot{\mathbf{H}}_0 \boldsymbol{\partial} \Delta w_\tau - \Delta \mathring{\boldsymbol{\varepsilon}}^{\cdot *}, \qquad \tau = 0, 1, \ldots, \tag{2.25}$$

the tensors of the initial states: $\Delta \mathring{\boldsymbol{\varepsilon}}^*$ and $\Delta \mathring{\boldsymbol{\varepsilon}}^{\cdot *}$ being determined only for $t = 0$

$$\Delta \mathring{\boldsymbol{\varepsilon}}^* = \sum_r \Delta \mathbf{C}^r \Delta \mathring{\varepsilon}^r,$$

$$\Delta \dot{\mathring{\boldsymbol{\varepsilon}}}^* = \sum_r \frac{\dot{\gamma}_0^r}{\gamma_0^r} \Delta \mathbf{C}^r \Delta \mathring{\varepsilon}^r + \widehat{\mathbf{K}}_0^* \Delta \mathring{\boldsymbol{\varepsilon}}^*,$$

$$\Delta \dot{\mathbf{H}}_0 = \sum_r \frac{\dot{\gamma}_0^r}{\gamma_0^r} z^r \Delta \mathbf{C}^r (\mathbf{T}_1^r)^T + \widehat{\mathbf{K}}_0^* \sum_r z^r \Delta \mathbf{C}^r (\mathbf{T}_1^r)^T,$$

$$\widehat{\mathbf{K}}_0^* = h \mu_0^2 \widehat{\mathbf{K}}_0^{-1} \tilde{\boldsymbol{\eta}}_0 - \sum_r \frac{\dot{\gamma}_0^r}{\gamma_0^r} \Delta \mathbf{C}^r (\mathbf{T}_1^r)^T.$$

Elements of the matrices $\Delta \mathbf{C}^r$ and $\Delta \widehat{\mathbf{K}}_0$ are the same as in (2.10) and (2.11).

The stepwise increments of moments and rates of moments in the plane of the plate are defined as it was done for (2.7). Thus we find:

$$\Delta \mathbf{M}_\tau = \mathbf{D}_0 \boldsymbol{\partial} \Delta w_\tau - \Delta \mathring{\mathbf{M}}^*, \tag{2.26}$$

$$\Delta \dot{\mathbf{M}}_\tau = \mathbf{D}_0 \boldsymbol{\partial} \Delta \dot{w}_\tau + \dot{\mathbf{D}}_0 \boldsymbol{\partial} \Delta w_\tau - \Delta \mathring{\mathbf{M}}^{\cdot *}, \tag{2.27}$$

where $\mathbf{D}_0$ and $\dot{\mathbf{D}}_0$ are matrices of instantaneous elasticity,

$$\Delta\overset{\circ}{\mathbf{M}}{}^{*} = \sum_r \Delta\overset{\circ}{\mathbf{S}}{}^{r} z^r - \sum_r \frac{A^r}{b^r\gamma_0^r} z^r \mathbf{T}_1^r(\mathbf{T}_1^r)^T \Delta\overset{\circ}{\boldsymbol{\varepsilon}}{}^{*},$$

$$\Delta\overset{\circ}{\mathbf{M}}{}^{\cdot *} = -\sum_r \Delta\overset{\circ}{\mathbf{S}}{}^{\cdot r} z^r + \sum_r \frac{A^r\dot{\gamma}_0^r}{b^r(\gamma_0^r)^2} z^r \mathbf{T}_1^r(\mathbf{T}_1^r)^T \sum_s \Delta\mathbf{C}^s \Delta\overset{\circ}{\varepsilon}{}^{s}$$
$$+ \sum_r \frac{A^r}{b^r\gamma_0^r} z^r \mathbf{T}_1^r(\mathbf{T}_1^r)^T \left[\sum_s \frac{\dot{\gamma}_0^s}{\gamma_0^s}\Delta\mathbf{C}^s\Delta\overset{\circ}{\varepsilon}{}^{s} + \widehat{\mathbf{K}}_0^* \sum_s \Delta\mathbf{C}^s\Delta\overset{\circ}{\varepsilon}{}^{s}\right],$$

$$\Delta\overset{\circ}{\mathbf{S}}{}^{r} = \frac{A^r}{b^r\gamma_0^r}\mathbf{T}_1^r\Delta\overset{\circ}{\varepsilon}{}^{r}, \qquad \Delta\overset{\circ}{\mathbf{S}}{}^{\cdot r} = \frac{\dot{\gamma}_0^r}{\gamma_0^r}\Delta\overset{\circ}{\mathbf{S}}{}^{r}.$$

The increments of shear forces and shear force rates will be determined from the formulae

$$\Delta\mathbf{T}_\tau = \boldsymbol{\partial}_1\Delta\mathbf{M}_\tau, \qquad \Delta\dot{\mathbf{T}}_\tau = \boldsymbol{\partial}_1\Delta\dot{\mathbf{M}}_\tau. \tag{2.28}$$

Knowing the constitutive equations which have already been derived, we can easily obtain, for any instant of time, three equations, the unknowns being the deflection of the plate $w_\tau(x,y)$, the stepwise increment of deflection $\Delta w_\tau(x,y)$ and the stepwise increment of the deflection rate $\Delta\dot{w}_\tau(x,y)$. Thus, three problems must be solved, in general, for every instant of time t_τ. It is assumed that the plate remains undeformed for $t < 0$. For $t = 0$ distortions are introduced into the fibre phase $\Delta\overset{\circ}{\varepsilon}$, and the plate is loaded $(\Delta p_0, \Delta\dot{p}_0)$, therefore it is necessary to determine Δw_0 and $\Delta\dot{w}_0$. At any subsequent instant of time t_τ $(\tau = 1,2,\ldots)$ we determine w_τ. If $\Delta p_\tau \neq 0$ or $\Delta\dot{p}_\tau \neq 0$, the quantities Δw_τ and $\Delta\dot{w}_\tau$ must also be determined.

3. The equations of the Finite Elements Method

It is assumed that the plate is acted on by a vertical load of intensity $p(\mathbf{x},t)$. According to the approximation which has been used with respect to the time, this load is represented by a sequence of functions $p_\tau(\mathbf{x})$, $\tau = 1,2,\ldots$. The load and the load rate may undergo, at selected instants of time, stepwise increments $\Delta p_\tau(\mathbf{x})$ and $\Delta\dot{p}_\tau(\mathbf{x})$.

The boundary conditions should be referred to the deflections and the forces at any time t_τ, to the stepwise increments of deflection and force and stepwise increments of deflection and force rates at the discontinuity points p_τ and $\dot{p}_\tau$ on the time axis. If the boundary conditions are of a static type, the prescribed quantities are the vectors of generalized boundary forces: $\overline{\mathbf{R}}_\tau$,

$\Delta\overline{\mathbf{R}}_\tau$ and $\Delta\dot{\overline{\mathbf{R}}}_\tau$. If they are of the kinematic type, we define the generalized boundary conditions using: $\overline{\mathbf{r}}_\tau$, $\Delta\overline{\mathbf{r}}_\tau$ and $\Delta\dot{\overline{\mathbf{r}}}_\tau$.

Under a load, the plate undergoes deflection $w(\mathbf{x},t)$ represented by the series of $w_\tau(\mathbf{x})$, of the increments of deflection being $\Delta w_\tau(\mathbf{x})$ and of the deflection rate $\Delta\dot{w}_\tau(\mathbf{x})$. The vector normal to the middle plane undergoes the rotations $\theta_{x,\tau}(\mathbf{x})$ and $\theta_{y,\tau}(\mathbf{x})$, the increments of rotation being $\Delta\theta_{x,\tau}(\mathbf{x})$ and $\Delta\theta_{y,\tau}(\mathbf{x})$ and increments of rotation rate $\Delta\dot{\theta}_{x,\tau}(\mathbf{x})$ and $\Delta\dot{\theta}_{y,\tau}(\mathbf{x})$.

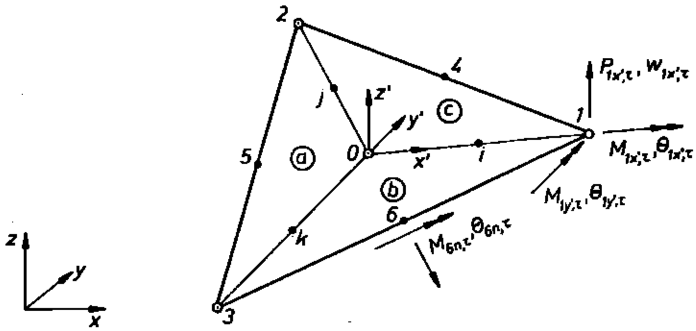

FIG. 2.

The problem is solved by using a suitable triangular element as shown in Fig. 2. The macroelement (Δ_{123}) is composed of three subelements denoted by the letters a, b and c. The origin of the system of local coordinates will be located at the gravity centre of the macroelement. The displacement fields: w^a_τ, w^b_τ and w^c_τ in the relevant subelements are approximated by a complete cubic polynomial:

$$w^a_\tau = \mathbf{p}^T\mathbf{a}_\tau\,, \qquad w^b_\tau = \mathbf{p}^T\mathbf{b}_\tau\,, \qquad w^c_\tau = \mathbf{p}^T\mathbf{c}_\tau\,, \tag{3.1}$$

where

$$\begin{aligned}\mathbf{p} &= \mathrm{col}(1, x', y', x'^2, x'y', y'^2, x'^3, x'^2y', x'y'^2, y'^3),\\ \mathbf{a}_\tau &= \mathrm{col}(a_{1,\tau}, a_{2,\tau}, a_{3,\tau}, \ldots, a_{9,\tau}, a_{10,\tau}) \quad \text{etc.}\end{aligned}$$

The fields of displacement increment: Δw^a_τ, Δw^b_τ, Δw^c_τ and displacement rate increments: $\Delta\dot{w}^a_\tau$, $\Delta\dot{w}^b_\tau$, $\Delta\dot{w}^c_\tau$ are approximated in a similar manner.

Thus, the deflection w_τ in a macroelement is described in terms of 30 parameters. Making use of the compatibility conditions for the generalized displacements at the central node we eliminate 6 parameters. The remaining compatibility conditions for principal and intermediate nodes of the macroelement and the subelements lead to a matrix equation, which reduces to the following form [20]

$$\tilde{\mathbf{a}}^e_\tau = \mathbf{A}^e_{a0}\mathbf{r}^e_\tau\,, \tag{3.2}$$

where $\mathbf{A}^e_{a0}$ is a matrix ensuring continuity of the approximation and its normal derivatives at the boundaries between the subelements,

$$\mathbf{r}^e_\tau = \mathrm{col}(w_{1,\tau}, \theta_{1x,\tau}, \theta_{1y,\tau}, w_{2,\tau}, \theta_{2x,\tau}, \theta_{2y,\tau}, w_{3,\tau}, \theta_{3x,\tau}\theta_{3y,\tau}, \theta_{4n,\tau}, \theta_{5n,\tau}, \theta_{6n,\tau}),$$
$$\widetilde{\mathbf{a}}^e_\tau = \mathrm{col}(a_{1,\tau}, a_{2,\tau}, \ldots, b_{3,\tau}, b_{4,\tau}, \ldots, c_{3,\tau}, c_{4,\tau}, \ldots, c_{9,\tau}, c_{10,\tau}).$$

The equation of virtual work is expressed at the level of a finite element

$$\text{(3.3)} \qquad \iint\limits_{\Omega^e} \delta(\boldsymbol{\partial} w_\tau)^T \mathbf{M}_\tau \, d\Omega = \iint\limits_{\Omega^e} \delta w_\tau p_\tau \, d\Omega + \int\limits_{\partial\Omega^e} \delta \overline{\mathbf{r}}^T_\tau \overline{\mathbf{R}}_\tau \, d(\partial\Omega).$$

This equation is written for every element, for which the constitutive equations (which have been derived) are taken into consideration as well as the relations resulting from the approximation assumed. For the subelement a we have

$$\text{(3.4)} \qquad \delta\mathbf{a}^T_\tau \iint\limits_{\Omega^a} (\boldsymbol{\partial}\mathbf{p}^T)^T \mathbf{D}_\tau \boldsymbol{\partial}\mathbf{p}^T \, d\Omega \, \mathbf{a}_\tau = \delta\mathbf{a}^T_\tau \Bigg[\iint\limits_{\Omega^a} \mathbf{p} p_\tau \, d\Omega$$
$$+ \int\limits_{\partial\Omega^a} \overline{\mathbf{p}}\overline{\mathbf{R}}_\tau \, d(\partial\Omega) + \iint\limits_{\Omega^a} (\boldsymbol{\partial}\mathbf{p}^T)^T \mathbf{M}^*_{\tau-1} \, d\Omega + \iint\limits_{\Omega^a} (\boldsymbol{\partial}\mathbf{p}^T)^T \, \mathring{\mathbf{M}}^* \, d\Omega \Bigg].$$

The equations for the elements b and c will be formulated in the same way.

By aggregating the subelements and taking into consideration the relation (3.2), we obtain the following equations of the finite elements method for a macroelement in the local system of coordinates

$$\text{(3.5)} \qquad (\mathbf{K}'_\tau)^e \mathbf{r}^e_\tau = (\mathbf{P}'_\tau)^e + (\overline{\mathbf{P}}'_\tau)^e + (\overline{\mathbf{P}}'^*_{\tau-1})^e + (\mathring{\mathbf{P}}'^*)^e,$$

where

$$(\mathbf{K}'_\tau)^e = (\mathbf{A}^T_{a0})^e (\widetilde{\mathbf{K}}'_\tau)^e (\mathbf{A}_{a0})^e,$$
$$(\mathbf{P}'_\tau)^e = (\mathbf{A}^T_{a0})^e (\widetilde{\mathbf{P}}'_\tau)^e, \qquad (\overline{\mathbf{P}}'_\tau)^e = (\mathbf{A}^T_{a0})^e (\widetilde{\overline{\mathbf{P}}}'_\tau)^e \qquad \text{etc.}$$

The symbol $(\mathbf{K}'_\tau)^e$ in Eq. (3.5) denotes the stiffness matrix of the macroelement, $(\mathbf{P}'_\tau)^e$ and $(\overline{\mathbf{P}}'_\tau)^e$ are load vectors, $(\mathbf{P}'^*_{\tau-1})^e$ is a vector expressing the state of the system at the preceding instant of time, and $(\mathring{\mathbf{P}}'^*_\tau)^e$ is a vector of distortion states in the macroelement. The tilde ~ denotes the corresponding matrices before static condensation.

Expressions for the stiffness matrix of the macroelement and the load vectors in the global system will be obtained by the following transformations

$$\text{(3.6)} \qquad \mathbf{K}^e_\tau = (\mathbf{T}^e)^T (\mathbf{K}'_\tau)^e (\mathbf{T}^e), \qquad \mathbf{P}^e_\tau = (\mathbf{T}^e)^T (\mathbf{P}'_\tau)^e \qquad \text{etc.}$$

The global matrix of viscoelastic stiffness of the plate $\mathbf{K}_\tau$ and the global vectors of load are constructed according to the general principles of the finite elements method. For the entire system we finally obtain the equation

$$\mathbf{K}_\tau \mathbf{r}_\tau = \mathbf{P}_\tau + \overline{\mathbf{P}}_\tau + \mathbf{P}^*_{\tau-1} + \overset{\circ}{\mathbf{P}}{}^*, \tag{3.7}$$

with which the kinematic boundary conditions must be satisfied. From (3.7) we obtain the global vector of generalized displacements $\mathbf{r}_\tau$.

At the discontinuity points of the load vector or the load rate vector on the time axis there occur jump-like variations of the internal forces and strains and their rates, both in the matrix and in the fibres. Those variations are caused by, among other factors, prestressing. In this connection we must determine the vectors of increment of nodal displacements $\Delta \mathbf{r}_\tau$ and nodal displacement rates $\Delta \dot{\mathbf{r}}_\tau$.

Equations of the finite elements method for $\Delta \mathbf{r}_\tau$ and $\Delta \dot{\mathbf{r}}_\tau$ can be derived in a manner analogous to that described above. For this aim the following equations are used:

$$\iint_{\Omega^e} \delta(\boldsymbol{\partial} \Delta w_\tau)^T \Delta \mathrm{M}_\tau \, d\Omega = \iint_{\Omega^e} \delta \Delta w_\tau \Delta p_\tau \, d\Omega + \int_{\partial\Omega^e} \delta \Delta \overline{\mathbf{r}}_\tau^T \Delta \overline{\mathbf{R}}_\tau \, d(\partial\Omega), \tag{3.8}$$

$$\iint_{\Omega^e} \delta(\boldsymbol{\partial} \Delta \dot{w}_\tau)^T \Delta \dot{\mathrm{M}}_\tau \, d\Omega = \iint_{\Omega^e} \delta \Delta \dot{w}_\tau \Delta \dot{p}_\tau \, d\Omega + \int_{\partial\Omega^e} \delta \Delta \dot{\overline{\mathbf{r}}}_\tau^T \Delta \dot{\overline{\mathbf{R}}}_\tau \, d(\partial\Omega), \tag{3.9}$$

as well as the constitutive equations (2.26) and (2.27) and the aproximation relations. Equations (3.9) are obtained from the equations of equilibrium expressed in terms of rates by performing typical operations, use being made of the boundary and geometrical equations.

On aggregating the entire system we obtain:

$$\begin{aligned} \mathbf{K}_0 \Delta \mathbf{r}_0 &= \Delta \mathbf{P}_0 + \Delta \overline{\mathbf{P}}_0 + \Delta \overset{\circ}{\mathbf{P}}{}^*, \\ \mathbf{K}_0 \Delta \dot{\mathbf{r}}_0 &= \Delta \dot{\mathbf{P}}_0 + \Delta \dot{\overline{\mathbf{P}}}_0 - \dot{\mathbf{K}}_0 \Delta \mathbf{r}_0 + \Delta \overset{\circ}{\mathbf{P}}{}^{\cdot *}, \qquad \tau = 0, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \mathbf{K}_0 \Delta \mathbf{r}_\tau &= \Delta \mathbf{P}_\tau + \Delta \overline{\mathbf{P}}_\tau, \\ \mathbf{K}_0 \Delta \dot{\mathbf{r}}_\tau &= \Delta \dot{\mathbf{P}}_\tau + \Delta \dot{\overline{\mathbf{P}}}_\tau - \dot{\mathbf{K}}_0 \Delta \mathbf{r}_\tau, \qquad \tau = 1, 2, \ldots . \end{aligned} \tag{3.11}$$

After determining the global vector of displacement $\mathbf{r}_\tau$ from the (3.7), the global vector of increment of displacement $\Delta \mathbf{r}_\tau$ and the global vector of increment of the displacement rate $\Delta\dot{\mathbf{r}}_\tau$ (from (3.10) and (3.11)), we must determine, for each subelement, the tensor fields of strain, the increment of strains and the increments of rates of those tensor fields. Making use of the physical equations which have already been derived we calculate the internal forces in the matrix and in the fibres, the increments of these forces and the stepwise variations of their rates.

The equations of the method of finite elements, the constitutive relations and the computation procedure described above enable us to determine the state of the system at any instant of time t_τ ($\tau = 0, 1, 2, \ldots$). After solving the set of equations presented above it is possible to proceed to consider the next instant of time $t_{\tau+1}$, for which a similar problem must be solved. Initial conditions for t_0 must be formulated to determine the left-hand limit of the vector fields of displacements and their rates.

The theory discussed above and the equations of the finite elements method have been used to develop a computer program for solving problems of prestressed viscoelastic plates of any form, subjected to any external load.

4. Examples of analysis

4.1. Example 1

The subject of this analysis is the behaviour of a rectangular diphase plate as represented in Fig. 3. The plate is acted on by a uniformly distributed load $p_\tau = 10\,\mathrm{kN/m^2}$ applied at the instant $t = 0^+$, and then held constant. The dead weight of the plate (g) is taken into consideration.

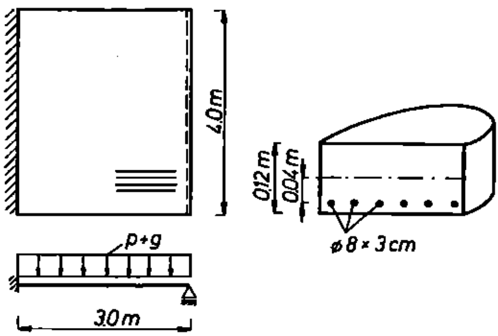

Fig. 3.

The plate has been prestressed by steel fibres $\phi = 0.008\,\mathrm{m}$ in diameter. It is assumed that the fibres have been initially tensioned with a force $\overset{\circ}{S}$, thus producing, after the joining process with the matrix is finished, a state of initial strain. The numerical analysis was made assuming various values for the force $\overset{\circ}{S}$ (0 to 10 kN).

The rheological parameters of the matrix were assumed according to the data contained in the monograph of KISIEL [21] and concerning the investigation by Mitzel and Dziendziel, who determined the viscoelastic parameters of concrete: $E_1 = 3.5 \cdot 10^4$ MPa, $E_2 = 9.5 \cdot 10^3$ MPa, $\eta_2 = 4.992 \cdot 10^6$ MPah.

It was assumed that the stress and strain deviator tensors are interrelated by the following differential relation of the Zener model of the first kind

$$\sigma_{ij,\tau} + \frac{\eta_2}{\mu_1 + \mu_2}\dot{\sigma}_{ij,\tau} = 2\frac{\mu_1\mu_2}{\mu_1 + \mu_2}\varepsilon_{ij,\tau} + 2\frac{\mu_1\eta_2}{\mu_1 + \mu_2}\dot{\varepsilon}_{ij,\tau}, \tag{4.1}$$
$$\mu_i = \frac{E_i}{2(1+\nu_i)}, \qquad i = 1,2,$$

where η_2 is the coefficient of shear viscosity and μ_i $(i = 1,2)$ is the shear modulus.

As regards the spherical parts of the tensors it was assumed that a viscoelastic body behaves, under hydrostatic tension or compression, in the same manner as an elastic body, therefore we have the following relation

$$\sigma_{kk,\tau} = 3K\varepsilon_{kk,\tau}, \tag{4.2}$$

where $K = E/(3(1-2\nu))$. It was assumed for computation that $E = E_1 = E_2 = 3.5 \cdot 10^4$ MPa, $\nu = \nu_1 = \nu_2 = 0.18$. The properties of the fibres were described by an equation of the Hooke model ($E^r = 2.05 \cdot 10^5$ MPa). Numerical analysis was made for a time interval $\langle 0, 3000\,\text{h}\rangle$ with a step $\vartheta_\tau = 120$ h. Discretization of the plate in space was performed in agreement with Fig. 4.

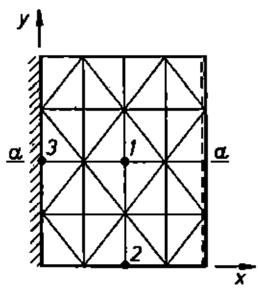

FIG. 4.

The influence of the prestressing force on the results of computation is illustrated by an example of the plate deflections at the points 1 (solid line) and 2 (dashed line) (Fig. 5), and by the variation of the distribution of the bending moments in the section $\alpha - \alpha$ (Fig. 6). The moments were obtained as mean values of the moments occuring in adjacent elements. The influence of the prestressing force on the values of M_x at the points 1 and 3 (Fig. 7) was also studied. Similarly to the case of deflection (Fig. 5), the relation between $\mathring{S}$ and M_x^1 and M_x^3 was found to be linear.

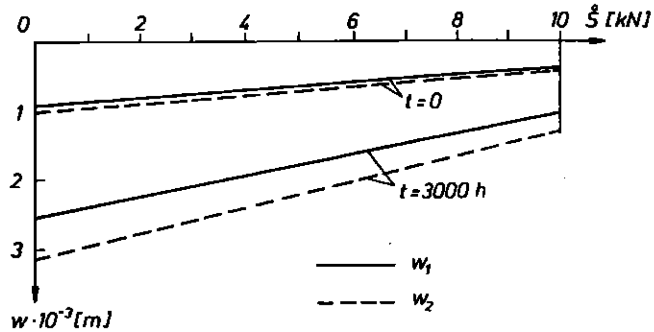

FIG. 5.

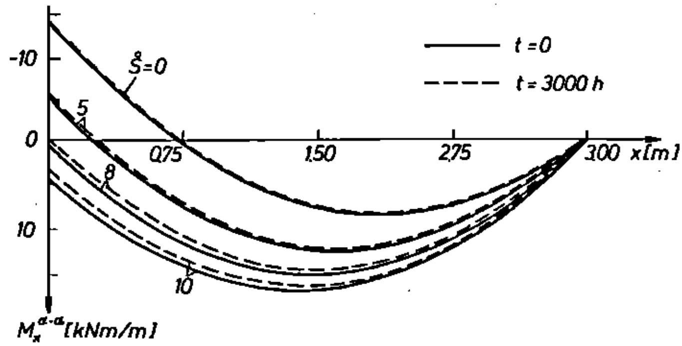

FIG. 6.

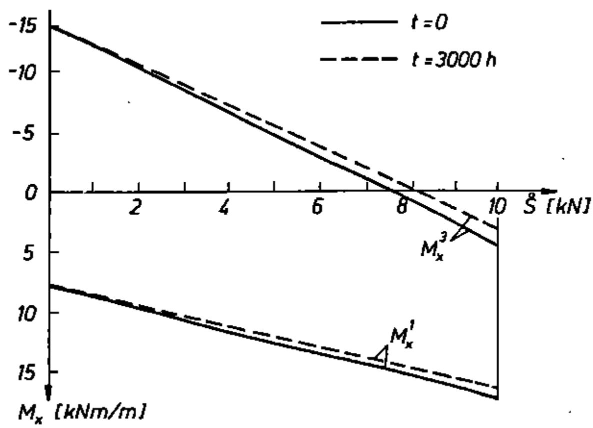

FIG. 7.

The time dependence was determined for the deflection at the point 2. Figure 8 shows essential changes in displacement for the prestressed plate. The results for a non-prestressed fibre-reinforced plate ($\mathring{S} = 0$) are also presented.

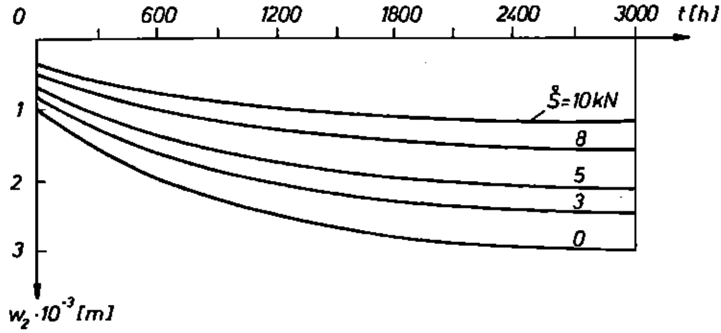

FIG. 8.

Another subject of the analysis was the time variation of the bending moments. Taking as an example the moment M_x^3, it is shown that those variations are insignificant, in particular if we are concerned with a non-prestressed fibre-reinforced plate (Fig. 9).

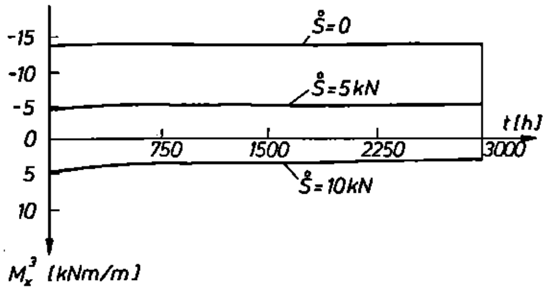

FIG. 9.

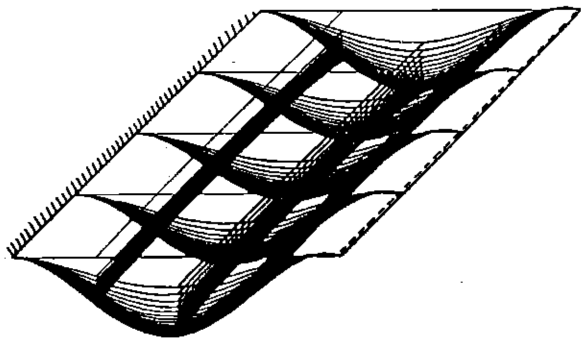

FIG. 10.

Finally, the time variation of the form of the plate prestressed by a force $\overset{\circ}{S} = 8\,\mathrm{kN}$ are presented in Fig. 10. The scale of the drawing and that of the displacements are different.

4.2. Example 2

A rectangular diphase plate, the dimensions of which are shown in Fig. 11, is reinforced in its upper part with two families of fibres ($\phi = 0.008\,\mathrm{m}$) in planes parallel to the middle plane of the plate, their directions being parallel to the edges of the plate. It is assumed that the fibres belonging to the family I may be prestressed with a force $\overset{\circ}{S}{}^{\mathrm{I}}$ and those of the family II – with a force $\overset{\circ}{S}{}^{\mathrm{II}}$. The analysis was made for various values of $\overset{\circ}{S}{}^{\mathrm{I}}$ and $\overset{\circ}{S}{}^{\mathrm{II}}$ (0 to 8 kN), for a period of 1200 h, by steps $\vartheta_\tau = 120\,\mathrm{h}$. It was assumed that $p_\tau = 6\,\mathrm{kN/m^2} = \mathrm{const}$. The rheological parameters of the matrix and the fibres were the same as in the Example 1.

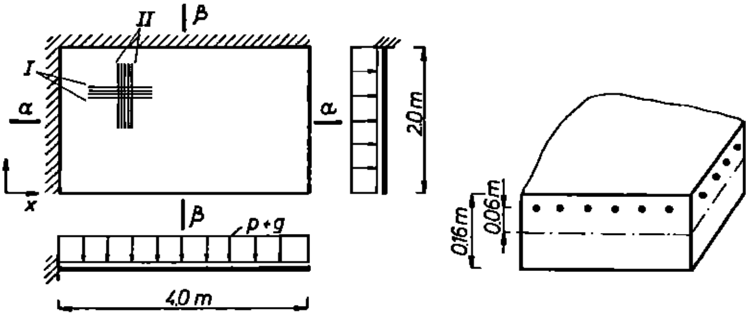

FIG. 11.

The influence of the prestressing forces $\overset{\circ}{S}{}^{\mathrm{I}}$ and $\overset{\circ}{S}{}^{\mathrm{II}}$ on the deflection of the plate are illustrated taking as an example the deflection of the point 1. This deflection (denoted w_1) at the initial moment is shown in Fig. 12 as a function of two variables $\overset{\circ}{S}{}^{\mathrm{I}}$ and $\overset{\circ}{S}{}^{\mathrm{II}}$. The image of that function is found to be a plane. The variation of the deflection w_1 as a function $\overset{\circ}{S}{}^{\mathrm{I}}$ and $\overset{\circ}{S}{}^{\mathrm{II}}$ after a period of 1200 h is illustrated in a similar manner.

Another subject of the analysis was the influence of the prestress on the distribution of the bending moments and their variation in time. Let us consider the case in which the plate is prestressed in the direction II only ($\overset{\circ}{S}{}^{\mathrm{I}} = 0$). The distribution of the bending moment M_y in the section $\beta - \beta$ is shown in Fig. 13, diagrams of M_x in the section $\alpha - \alpha$ (normal to the direction of reinforcement) being shown in Fig. 14.

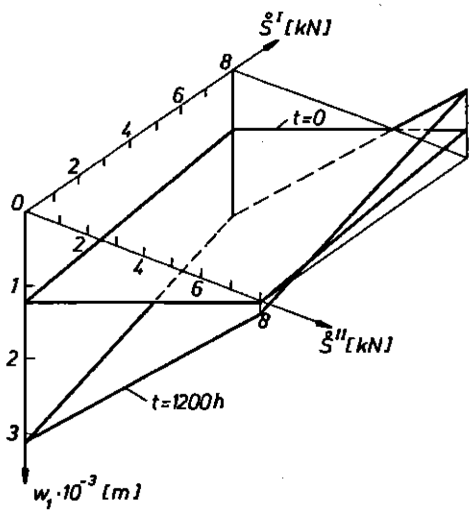

FIG. 12.

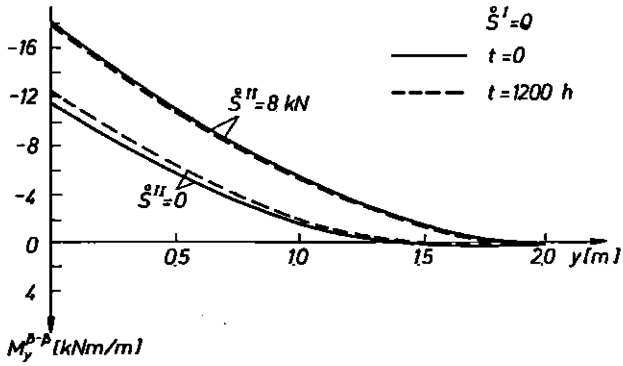

FIG. 13.

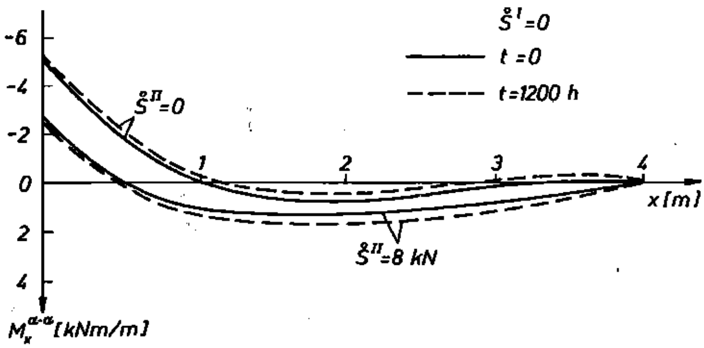

FIG. 14.

In the second case the plate was prestressed in the direction of the x-axis ($\overset{\circ}{S}{}^{\mathrm{II}} = 0$). The distribution of the moment M_y in the section $\beta - \beta$ and M_x in the section $\alpha - \alpha$, and their variation as functions of time, are shown in Figs. 15 and 16. Figure 16 shows that the influence of the force $\overset{\circ}{S}{}^{\mathrm{I}}$ on the value of M_x is considerable and is insignificant as regards the variation of the distribution of the moments M_y in the section $\beta - \beta$.

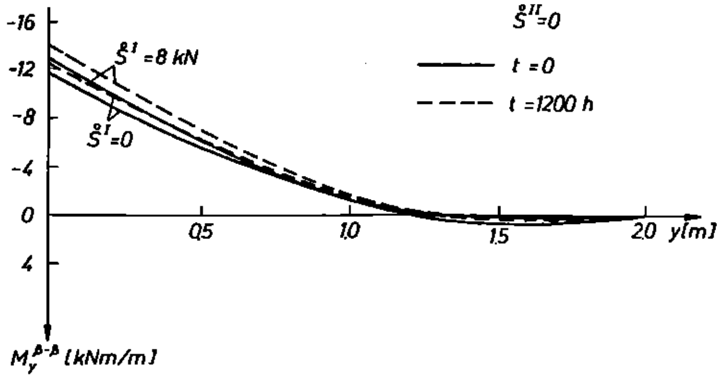

FIG. 15.

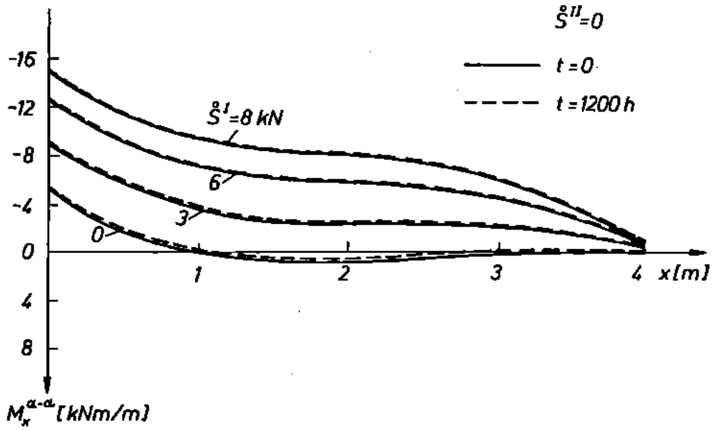

FIG. 16.

The variation of the deflection in time, depending on the prestress, is illustrated in Fig. 17. The solid line illustrates the variation of the displacement before the external load is removed, and the dashed line shows the results obtained after its removal for $t = 1200$ h.

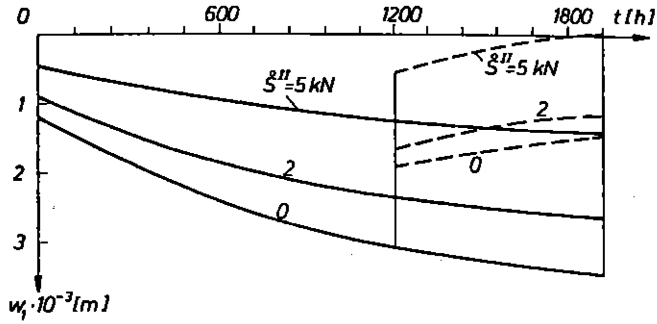

FIG. 17.

5. INFERENCES

The aim of the considerations presented above was to prepare an effective methed for analysis of the influence of prestressing and of the external load on the stress and strain in a diphase viscoelastic plate. Although the model assumed here introduces a limitation to the range of rheological problems considered to linear viscoelasticity (which is of course a simplification), its analysis may furnish much information essential for the estimation of the influence of the phenomenon of creep of particular phases on the behaviour of the system as a whole.

The present method of solution of that problem has been developed having in view the possibility of studying the rheological processes in plates from the moment of prestressing till their practical end.

The examples of solution illustrate the functioning of the algorithm and the computer program. Analyses of the results show that the influence of the rheological phenomena in prestressed plates with densely distributed fibres are essential. It has been shown that the method of prestressing is also essential for the creep process. The solutions obtained may be used for further analyses including optimization of the parameters for reinforced viscoelastic plates.

The diphase model of a solid is an idealization of the real structure. It follows that the computed distributions of displacements and internal forces and their variation in time constitute a certain approximation to real civil engineering problems (such as those of pretensioned prestressed concrete or plates of synthetic materials reinforced with glass, carbon or metal fibres). Thus, it is legitimate to state that the model used is more adequate for finer and more densely distributed fibres of the reinforcement.

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