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ELASTIC-PLASTIC PLATES SEALING A LIQUID; TESTS AND ANALYSIS

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Circular steel plates sealing the vessels containing compressible liquid are considered. The edge of the plate is built into the vessel wall. The plate is loaded at its central part and supported by the liquid pressure. The material is assumed to be elastic-perfectly plastic, obeying the Tresca yield criterion and the associated flow rule. A closed-form analytical solution is obtained in the case of monotonically increasing load. The corresponding experimental study is aimed at verification of the analytical approach. Special attention is given to the evolution of the apparent compressibility of the liquid due to air residuals and to the deformations of the lateral walls of the vessel.

Notation

Dimensional quantities

R [cm], θ	polar coordinates,
R_0 [cm]	plate radius,
A [cm]	load radius,
R_p [cm]	radius of plastic zone,
H [cm]	plate thickness,
$\eta = 2R_0/H$	plate slenderness,
ν	Poisson's ratio,
E [daN/cm^2]	Young's modulus,
E_l [daN/cm^2]	elastic bulk modulus,
D [daNcm]	plate stiffness,
W, W_o [cm]	deflection, central deflection,
k [daN/cm^3]	coefficient of the liquid (subgrade) reaction,
kW [daN/cm^2]	buoyancy pressure,
P [daN/cm^2]	distributed load,
F [T]	total load,
F_c [T]	total collapse load for plate sealing incompressible liquid,
Q [daN/cm^2 = bar]	unknown pressure of the sealing liquid,
ΔV	change of volume of sealed liquid due to its compressibility,
M_r, M_θ [daNcm/cm]	radial and circumferential moments,
S [daN/cm]	shear force,
K_r, K_θ [1/cm]	curvatures,
$(\dot{\ })$	denotes rate of an appropriate quantity.

Reference quantities

L [cm] reference length,
$L = \sqrt[4]{D/k}$ for plate resting on the elastic subgrade,
$L = R_o$ for plate without effect of buoyancy,
M_0 [daNcm/cm] plastic moment per unit width of the plate.

Dimensionless quantities

$$\varrho = \frac{R}{L}, \qquad \varrho_0 = \frac{R_0}{L}, \qquad a = \frac{A}{L}, \qquad \varrho_p = \frac{R_p}{L},$$

$$p = \frac{PL^2}{M_0}, \qquad q = \frac{QL^2}{M_0}, \qquad \overline{p} = \frac{\overline{P}L}{M_0},$$

$$m_i = \frac{M_i}{M_0}, \qquad i = r, \theta, 0, \qquad s = \frac{SL}{M_0},$$

$$w = \frac{D}{M_0 L^2} W, \qquad \kappa_i = K_i \frac{D}{M_0}, \qquad i = r, \theta,$$

$$\frac{W}{L} = \frac{M_0 L}{D} w, \qquad \frac{W}{H} = \frac{M_0 L^2}{DH} w, \qquad \frac{WH}{L^2} = \frac{HM_0}{D} w.$$

1. Introduction and statement of the problem

In this paper, the problem of describing the deformation process of a thin, circular plate sealing a compressible fluid is considered (see Fig. 1), and agreement of the theoretical solutions with the experimental results is studied.

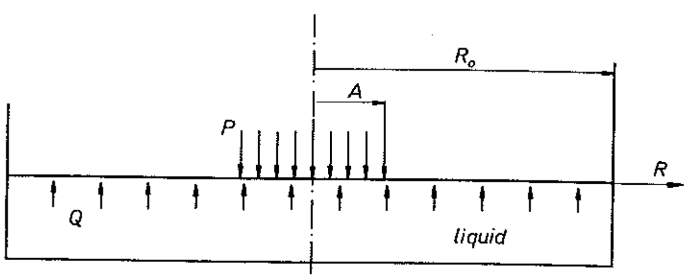


FIG. 1. Plate sealing a liquid.

Problems of circular plates resting on a continuous subgrade concerns the classical "soil-structure" interaction [8, 9], as well as floating structures and plates sealing a liquid in a constant-volume vessel. The last category of the problems has been treated until now either as elastic [2, 3, 4] or rigid-plastic [1, 6] ones.

In the present analysis the plate is assumed to be elastic-perfectly plastic, obeying Tresca's yield condition. The classical Kirchhoff–Love thin plates theory is followed at small deflections, and the plastic flow rule is used.

The hypothesis of lumped-type plastic deformation is assumed; cross-sections are considered to be either elastic or fully plastic. The above assumption is accurate in case of sandwich plates.

The external load uniformly distributed over a circular central region produces an unknown, uniformly distributed pressure Q at the liquid-plate interface.

The effect of buoyancy pressure is negligible as compared with the effect of the sealed liquid pressure [2, 3]. It is strictly nonexistent if the plate is covered additionally by a thin layer of the liquid.

Under the above assumptions, basic relations governing the plate behaviour, in terms of dimensionless quantities, are as follows:

Equilibrium equations:

$$(s\varrho)' + (p-q)\varrho = 0, \tag{1.1}$$

$$(m_r\varrho)' - m_\theta - s\varrho = 0, \tag{1.2}$$

geometrical relations:

$$\kappa_r = -w'', \qquad \kappa_\theta = -w'/\varrho, \tag{1.3}$$

curvature decomposition into the elastic and plastic parts:

$$\kappa_r = \kappa_r^e + \kappa_r^p, \qquad \kappa_\theta = \kappa_\theta^e + \kappa_\theta^p, \tag{1.4}$$

Hooke's law:

$$\kappa_r^e = \frac{1}{1-\nu^2}(m_r - \nu m_\theta), \qquad \kappa_\theta^e = \frac{1}{1-\nu^2}(m_\theta - \nu m_r), \tag{1.5}$$

the associated plastic flow rule:

$$\dot{\kappa}_r^p = \lambda\frac{\partial F}{\partial m_r}, \qquad \dot{\kappa}_\theta^p = \lambda\frac{\partial F}{\partial m_\theta}, \qquad \lambda > 0, \tag{1.6}$$

with the Tresca yield condition as a plastic potential, see Fig. 2

$$F = \sup(|m_r|,\ |m_\theta|,\ |m_r - m_\theta|) = m_0. \tag{1.7}$$

The plate displacements due to external load change the volume of the liquid sealed below the plate according to the formula

$$\Delta V = 2\pi\int_0^{R_0} WR\,dR, \tag{1.8}$$

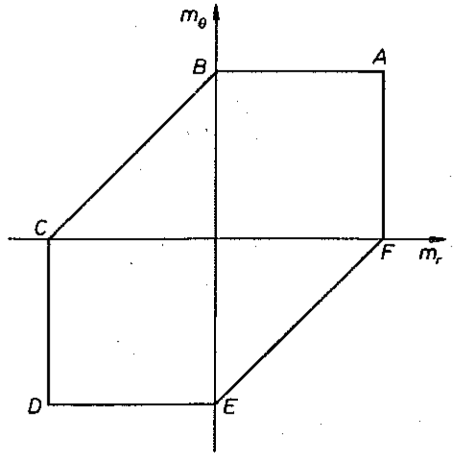


FIG. 2. The Tresca yield condition.

or, in dimensionless quantities,

(1.9)
$$\Delta \mathrm{v} = \Delta V D / M_0 L^4 = 2\pi \int_0^{\varrho_0} w \varrho \, d\varrho .$$

This volume may be either constant ($\Delta \mathrm{v} = 0$, incompressible fluid, nondeformable vessel walls), or its change may be a function of the liquid pressure. This function should be linear, since the liquid compressibility and elastic deformations of the vessel contribute to the volume changes:

(1.10)
$$\Delta V = \Phi Q, \qquad \left(Q = \frac{\Delta V}{\Phi} \right),$$

where $\Phi = V_i / E_l$, [$\mathrm{cm}^5/\mathrm{daN}$] is the coefficient of compressibility; V_i is the initial liquid volume, and E_l is the elastic bulk modulus.

If $\Phi \Rightarrow 0$ ($E_l \Rightarrow \propto$), then the liquid is incompressible,

if $\Phi \Rightarrow \propto$ ($E_l \Rightarrow 0$), then $Q \Rightarrow 0$,

(the plate deforms like a plate sealing an empty vessel).

In the case of linear compressibility, the condition (1.9) takes the form

(1.11)
$$\int_0^{\varrho_0} w \varrho \, d\varrho = \varphi q,$$

where
$$\varphi = \frac{1}{2\pi} \Phi \frac{D}{L^6}$$

is a non-dimensional coefficient of compressibility. In practice, when $\varphi > 3$, the liquid pressure can be neglected.

However, if a small amount of highly compressible gas is present (voids in the washer, the sealing device and the liquid-gas foam), these functions may be strongly nonlinear. Such a situation occurred in our experimental study and was confirmed by explosive behaviour that happened sometimes at rupture tests of containment structures loaded by oil pressure [7].

Thus the change of volume of the sealed liquid can be a nonlinear function of pressure,

$$\Delta V = f(Q). \tag{1.12}$$

A method worked out for plates resting on elastic subgrade [8, 9, 10] is used now to solve the problem of a plate sealing a compressible liquid. The governing differential equations and their solutions, as well as the analytical expresions for the fields of general stresses and strains for different zones, are collected in the Appendix.

The solution of a specific case must describe spreading of plastic zones and possible evolution of their configuration during the process of loading. At each qualitatively distinct stage of the loading process, the external boundary condition, together with the continuity requirements between different zones

$$w] = w'] = m_r] = m_\theta] = s] = 0, \tag{1.13}$$

have to be satisfied by the general solution.

Thus, a system of algebraic equations is obtained; they are linear with respect to the integration constants C_i and the pressure of the liquid q, but nonlinear in terms of the radii describing configuration of the zones.

2. Experimental tests

To compare the results of the above analysis, a short series of tests on steel models was performed at the Chair of Mechanics of Materials and Structures at the Faculté Polytechnique de Mons.

The specimens were cut from commercial hot-rolled steel and formed by turning to the needed shape (see Fig. 3). The material yield stress was $\sigma_0 = 210.2\,\text{MPa}$.

The test device consisted of a heavy steel backing ring, to which the plate specimen was clamped by means of bolts through a large clamping ring (see Fig. 4).

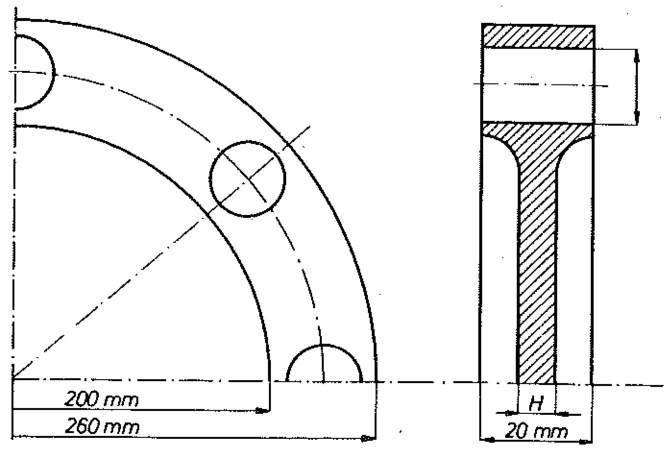


FIG. 3. Details of the tested plates.

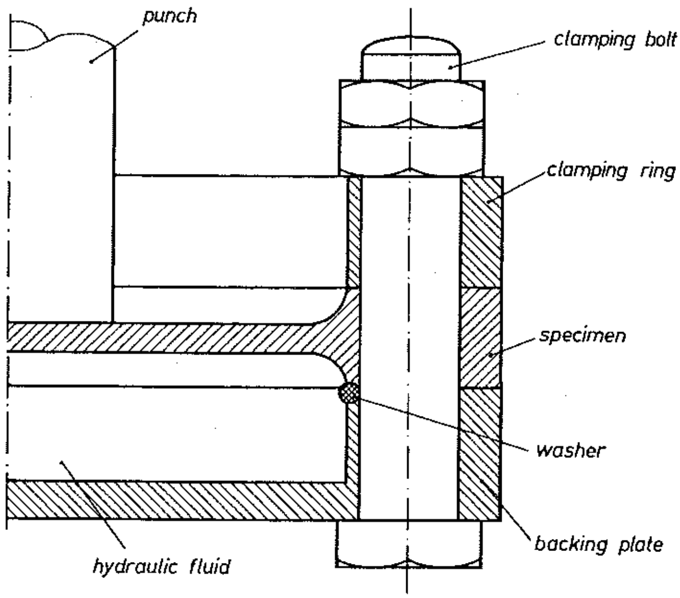


FIG. 4. Details of the test device.

The load was applied through a rigid central punch loaded by means of a universal testing machine. Deflections were measured by Hewlett Packard gauges 7DCDT500 of the sensitivity 3.848/Volt and in the range ±12.7 mm and recorded by digital voltmeter on an IBM/PC computer. Fluid pressure was monitored by a standard gauge of the sensitivity 29.93 bar/Volt and in the range 0÷150 bar.

Common assumptions of incompressibility or linear compressibility of the liquid were in disagreement with the results of experiments. Therefore it was necessary to conduct some tests describing the effective compressibility law of the sealed fluid (oil, air bubbles, volume changes), (see Fig. 5).

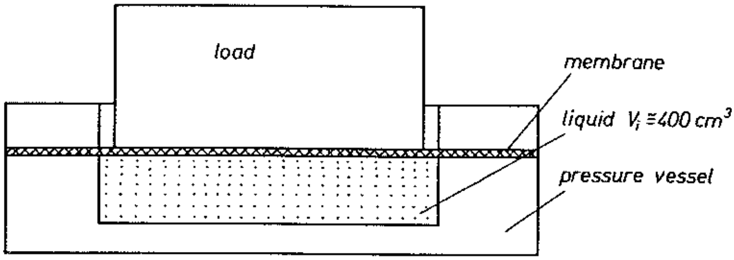


FIG. 5. Layout of the test used for calibrating compressibility of the fluid.

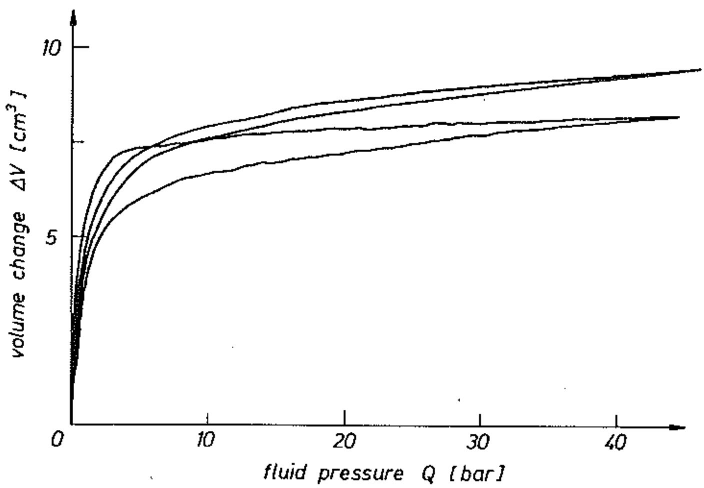


FIG. 6. Results of the effective compressibility tests.

The results of these simple tests are shown in Fig. 6. The shape of the volume changes versus pressure curve varied from test to test, depending on the initial volume of the air bubbles V_0. The changes of the fluid volume should be described as a sum of two components. The first one depends on the pressure and is significant only for high pressure. The second one depends on the air bubble volume in fluid and is important at the beginning of the process. The total volume change can be expressed as

$$\Delta V = \Phi Q + V_0 \frac{Q}{Q_0 + Q}, \tag{2.1}$$

where V_0 and Q_0 are initial air bubble volume and initial pressure, respectively. Figure 7 elucidates this relation. To describe compressibility of the fluid, it is necessary to find two parameters Φ and V_0. The first one depends upon a material constant and on the initial volume of the liquid sealed under a plate. The second one depends upon the unknown volume of air bubbles. They can be found analytically for a plate in elastic régime for external loads F and pressures Q_0, Q obtained from the test and the volume changes ΔV obtained from the elastic solutions.

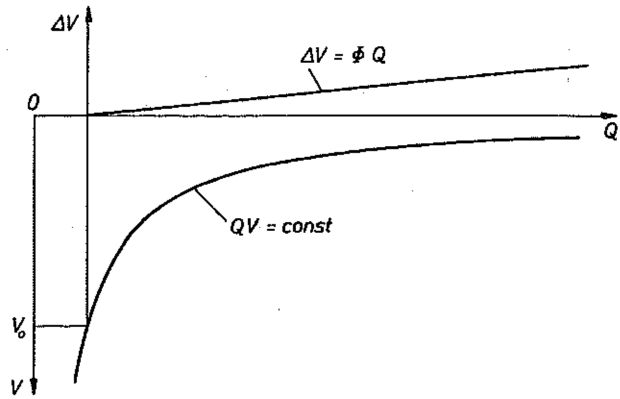


FIG. 7. Simplified description of compressibility of the fluid.

3. Results and conclusions

Three plates were tested having the same external radius $R_0 = 100$ mm and the same diameter of the loading punch $2A = 40$ mm. The thickness was $H = 4.37$ mm, 5.57 mm and 6.59 mm, corresponding to the slenderness ratio $\eta = 45$, 40 and 35.

Load versus punch deflection and load versus liquid pressure curves obtained from the tests are given in Figs. 8 and 9. Nearly linear parts of the load-deflection curves corresponding to elastic behaviour of the plate (beginning of the loading process, unloading, reloading) are not parallel. It is due, first of all, to the variation of the liquid effective compressibility, which is important at the first loading (low liquid pressure). For a more advanced deformation process, the compressibility is nearly constant and some steepening of the curves indicates rather a geometrical hardening due to large deformation.

Comparison of the test and theoretical results for the plate of $\eta = 45$ is given in Figs. 10 and 11.

The theoretical solutions for an incompressible liquid ($\Phi = 0$) and for linear compressibility ($\Phi = 0.1, 0.5, 1, 2, \infty$ cm^5/dN) presented in Figs. 10 and 11, are qualitatively different from the test results.

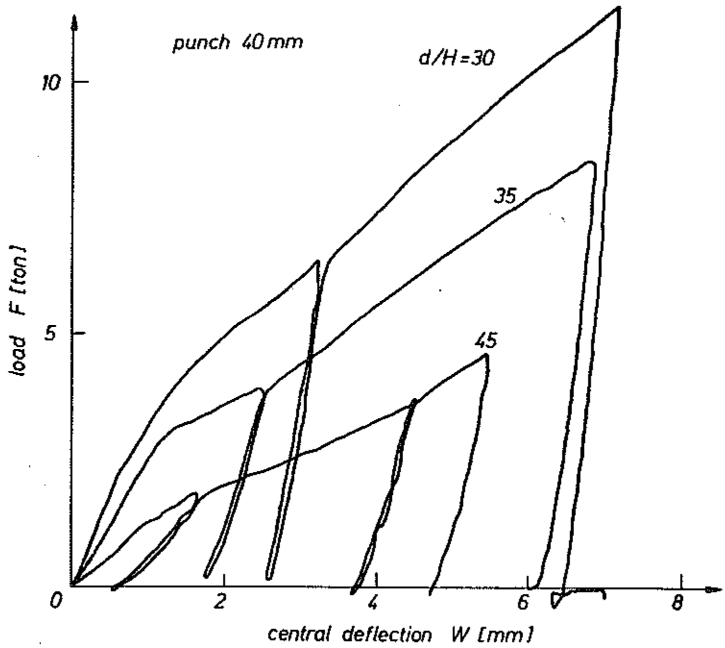


FIG. 8. Test results: load-deflection behaviour.

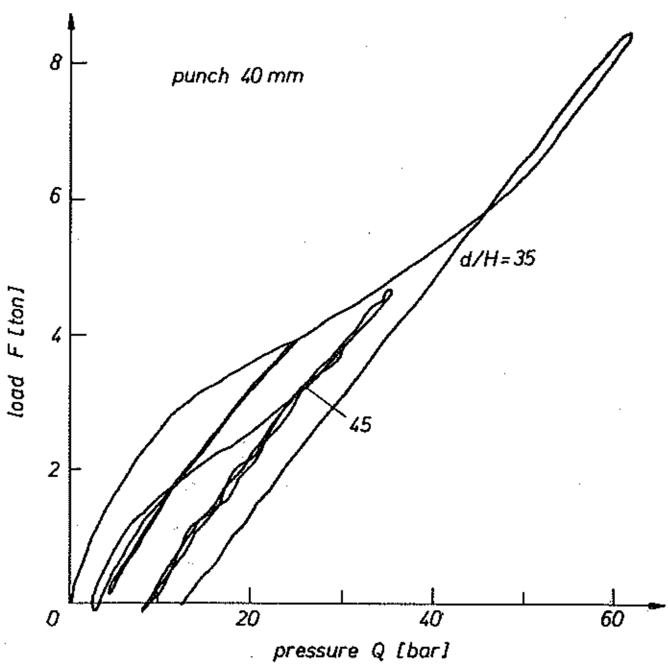


FIG. 9. Test results: load – liquid pressure relationship.

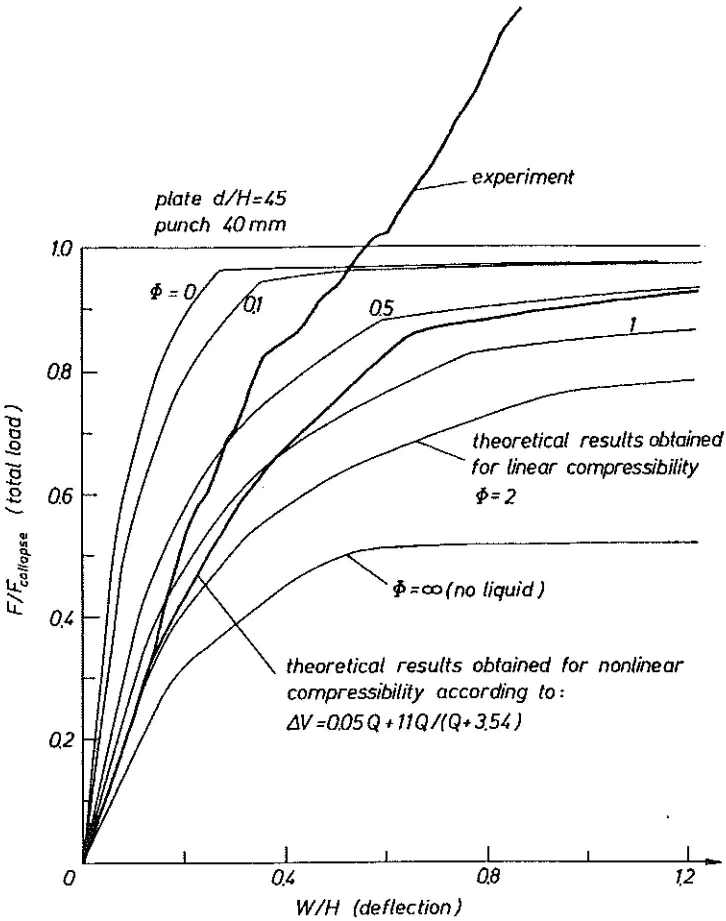


FIG. 10. Comparison of the theoretical and experimental results: load-deflection relationship.

Therefore the nonlinear compressibility law was applied, following Eq. (2.1). The parameters $\Phi = 0.05\,\mathrm{cm}^5/\mathrm{dN}$ and $V_0 = 11\,\mathrm{cm}^3$ were found from the requirement that the elastic solution complies with the corresponding part of the experimental curve. The coincidence is no more exact in the elastic-plastic phase, but is qualitatively acceptable. The inconsistency is due to membrane forces (geometrical hardening) and to the material hardening.

To conclude:

Confrontation of the experimental and theoretical data proved that the commonly accepted assumptions of either incompressibility or linear compressibility are far from the reality.

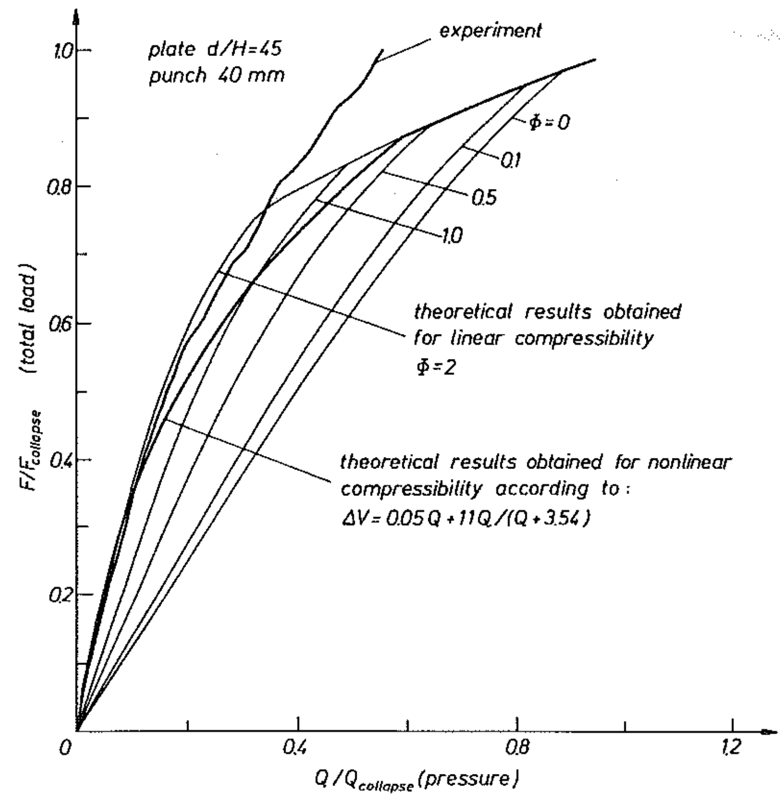


FIG. 11. Comparison of the theoretical and experimental results: load-liquid pressure relationship.

The theoretical aproach assuming the nonlinear compressible liquid (2.1) in the framework of the small deflection theory lead to acceptable results for displacements not exceeding one half of the thickness of the plate. However, for a more advanced deformation process, geometrical hardening should be taken into account.

APPENDIX

Zone elastic, [12]

Differential equation

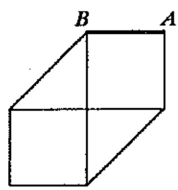
$$w^{\mathrm{IV}} + \frac{2}{\varrho} w^{\mathrm{III}} - \frac{1}{\varrho^2} w^{\mathrm{II}} + \frac{1}{\varrho^3} w^{\mathrm{I}} = p - q,$$

$$\int w\varrho \, d\varrho = C_1 \frac{\varrho^2}{2} + C_2 \frac{\varrho^4}{4} + C_3 \varrho^4 \left(\frac{\ln \varrho}{4} - \frac{1}{16} \right) + C_4 \varrho^2 \left(\frac{\ln \varrho}{2} - \frac{1}{4} \right) + \frac{p-q}{384} \varrho^6 .$$

Deflection, moments, shear force

$$
\begin{aligned}
w &= C_1 + C_2\varrho^2 + C_3\varrho^2 \ln\varrho + C_4 \ln\varrho + (p-q)\frac{\varrho^4}{64}\,,\\
w^{\mathrm{I}} &= 2C_2\varrho + C_3\varrho(1+2\ln\varrho) + C_4\frac{1}{\varrho} + (p-q)\frac{\varrho^3}{16}\,,\\
m_r &= -2C_2(1+\nu) - C_3(3+\nu+2(1+\nu)\ln\varrho) + C_4\frac{1-\nu}{\varrho^2} - (p-q)\frac{3+\nu}{16}\varrho^2,\\
m_\theta &= -2C_2(1+\nu) - C_3(3+3\nu+2(1+\nu)\ln\varrho) - C_4\frac{1-\nu}{\varrho^2} - (p-q)\frac{1+3\nu}{16}\varrho^2,\\
s &= -C_3\frac{4}{\varrho} - (p-q)\frac{\varrho}{2}\,.
\end{aligned}
$$

Zone AB; $\varrho = 0 \in AB$



Differential equation

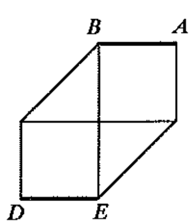
$$
w^{\mathrm{IV}} + \frac{2}{\varrho}w^{\mathrm{III}} = \frac{p-q}{1-\nu^2}\,,
$$

$$
\int w\varrho\,d\varrho = C_1\frac{\varrho^2}{2} + C_2\frac{\varrho^3}{3} - \frac{m_0}{8(1+\nu)}\varrho^4 + \frac{p-q}{432(1-\nu^2)}\varrho^6.
$$

Deflection, moments, shear force

$$
\begin{aligned}
w &= C_1 + C_2\varrho - \frac{m_0}{2(1+\nu)}\varrho^2 + \frac{p-q}{72(1-\nu^2)}\varrho^4,\\
w^{\mathrm{I}} &= C_2 - \frac{m_0}{1+\nu} + \frac{p-q}{18(1-\nu^2)}\varrho^3,\\
m_r &= m_0 - (p-q)\frac{\varrho^2}{6}\,,\\
m_\theta &= m_0\,,\\
s &= -(p-q)\frac{\varrho}{2}\,.
\end{aligned}
$$

Zone AB (DE)



Differential equation

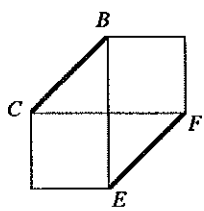
$$w^{\mathrm{IV}} + \frac{2}{\varrho} w^{\mathrm{III}} = \frac{p-q}{1-\nu^2},$$

$$\int w\varrho\, d\varrho = C_1 \frac{\varrho^2}{2} + C_2 \frac{\varrho^3}{3} + C_3 \frac{\varrho^4}{4} + C_4 \varrho^3 \left(\frac{\ln \varrho}{3} - \frac{1}{9} \right) + \frac{p-q}{432(1-\nu^2)} \varrho^6 .$$

Deflection, moments, shear force

$$\begin{aligned} w &= C_1 + C_2\varrho + C_3\varrho^2 + C_4 \varrho \ln \varrho + \frac{p-q}{72(1-\nu^2)} \varrho^4, \\ w^{\mathrm{I}} &= C_2 + 2C_3\varrho + C_4(1+\ln\varrho) + \frac{p-q}{18(1-\nu^2)}\varrho^3, \\ m_r &= \pm\nu m_0 - 2C_3(1-\nu^2) - C_4\frac{1-\nu^2}{\varrho} - (p-q)\frac{\varrho^2}{6}, \\ m_\theta &= \pm m_0, \\ s &= \mp\frac{1-\nu}{\varrho} m_0 - 2C_3\frac{1-\nu^2}{\varrho} - (p-q)\frac{\varrho}{2}. \end{aligned}$$

Zone BC (EF)



Differential equation

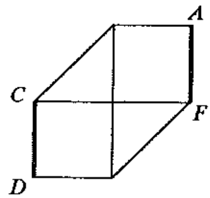
$$w^{\mathrm{IV}} + \frac{2}{\varrho} w^{\mathrm{III}} - \frac{1}{\varrho^2} w^{\mathrm{II}} + \frac{1}{\varrho^3} w^{\mathrm{I}} = (p-q)\frac{2}{1+\nu},$$

$$\int w\varrho\, d\varrho = C_1\frac{\varrho^2}{2} + C_2\frac{\varrho^4}{4} + C_3\varrho^4\left(\frac{\ln\varrho}{4} - \frac{1}{16}\right) + C_4\varrho^2\left(\frac{\ln\varrho}{2} - \frac{1}{4}\right) + \frac{p-q}{192(1+\nu)}\varrho^6.$$

Deflection, moments, shear force

$$\begin{aligned} w &= C_1 + C_2\varrho^2 + C_3\varrho^2\ln\varrho + C_4\ln\varrho + (p-q)\frac{\varrho^4}{32(1+\nu)}, \\ w^{\mathrm{I}} &= 2C_2\varrho + C_3\varrho(1+2\ln\varrho) + C_4\frac{1}{\varrho} + (p-q)\frac{\varrho^3}{8(1+\nu)}, \\ m_r &= \mp\frac{1}{2}m_0 - 2C_2(1+\nu) - 2C_3(1+\nu)(1+\ln\varrho) + C_4\frac{1-\nu}{\varrho^2} - (p-q)\frac{1}{4}\varrho^2, \\ m_\theta &= m_r \pm m_0, \\ s &= \mp\frac{1}{\varrho}m_0 - 2C_3\frac{1+\nu}{\varrho} - (p-q)\frac{\varrho}{2}. \end{aligned}$$

Zone CD (AF)



Differential equation

$$\frac{1}{\varrho^2}w^{\mathrm{II}} - \frac{1}{\varrho^3}w^{\mathrm{I}} = -(p-q)\frac{1}{1-\nu^2},$$

$$\int w\varrho\, d\varrho = C_1\frac{\varrho^2}{2} + C_2\frac{\varrho^4}{4} - \frac{p-q}{48(1-\nu^2)}\varrho^6.$$

Deflection, moments, shear force

$$\begin{aligned}
w &= C_1 + C_2\varrho^2 - (p-q)\frac{1}{8(1-\nu^2)}\varrho^4,\\
w^{\mathrm{I}} &= 2C_2\varrho - (p-q)\frac{1}{2(1-\nu^2)}\varrho^3,\\
m_r &= \mp m_0,\\
m_\theta &= \mp\nu m_0 - 2C_2(1-\nu^2) + (p-q)\frac{1}{2}\varrho^2,\\
s &= \mp\frac{1-\nu}{\varrho}m_0 + 2C_2(1-\nu^2)\frac{1}{\varrho} - (p-q)\frac{1}{2}\varrho.
\end{aligned}$$

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