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ANISOTROPY OF RANDOM FIELDS IN STOCHASTIC FEM

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The anisotropy of a random variable consisting of rectangular local averages of a Gaussian, isotropic and homogeneous random field is presented. The analysis deals with fields having exponential correlation functions. It is shown that such local averages, formed by a mesh of finite elements, generate the anisotropy even if the mesh consists of identical squares.

1. Introduction

The finite element method is a common tool in many different engineering calculations and is very useful to determine, for example, such values as displacements in solid mechanics or potentials in flow problems. Before modern computers were developed, the FEM calculations had to be deterministic because of low performance of most computers available then. However, real media, especially soils [1], exhibit randomness and so do the loads. Thus, deterministic calculations, though usually sufficient for engineers, do not fully describe the reality.

Nowadays, due to fast and high capacity computers, it was possible to develop several variants of the stochastic finite element analysis. The oldest approach known as the perturbation method [4] is still used in cases of small randomness. It is also suitable for the analysis of simple systems, for example a stochastic free-end beam. More sophisticated techniques such as the Neumann series expansion [5], the weighted integral method [6] or the orthogonal series expansion [7] often yield excellent results, but the experience proves that they are not universal. By contrast, the Monte Carlo method is universal and can be applied to any problem. In many cases it is the only method that can give satisfactory results, provided that the modelling of a specific system is accurate. Therefore the paper deals with some aspects of MC simulations in the stochastic FEM. It should however be realized that primitive formalism of the Monte Carlo simulations often results in unacceptably long computer runs, and other techniques are usually preferred to MC simulations, treated as the last resort.

The first step in each stochastic FEM application that must be done is to discretize the parameter space of random fields of material properties and, if necessary, the loads. In other words, the initial random fields are transformed into equivalent finite random variables. Some aspects of such a transformation are still not sufficiently known. One of them is anisotropy of the equivalent random variable. Our analysis is concentrated on one of the most common models – a Gaussian, homogeneous, isotropic random field with an exponential correlation function.

2. Point discretization of random field

Two-dimensional media are usually discretized by a mesh consisting of triangular and quadrilateral elements. The net is then refined until the results in two consecutive iterations differ little enough. This is typical of deterministic FEM and it is assumed here that the mesh which is deterministically optimum is also stochastically pertinent.

The simplest discretization method of the parameter space of random field is when the shape and size of elements are neglected. Since the field is homogeneous, its mean value does not change. Furthermore, the variance σ_u^2 of the random field averaged over the element is equal to the field point variance σ_p^2. Consequently, the covariance c_u between each pair of elements (local averages) is equal to the covariance c_p between the centroids of both elements. To summarize, we can say that such a discretization retains the isotropy but completely disregards the variability of correlation across the elements. This conclusion immediately indicates the main shortcoming of the method: it can be applied to cases in which the correlation properties change very little inside the elements. If these changes are higher, then the mesh must be refined to such an extent that the computer runs become unacceptably long.

3. Local averages of random fields

Theoretical bases of local averages of random fields can be found in [2, 3] and [6]. However, only raw formulas were derived there and few remarks made on the behaviour of the resulting finite random variable. That is why we briefly describe the principles of local averages, focusing on the anisotropy.

Let us take the Gaussian, homogeneous, isotropic random field $f(x,y)$. Its parameters are as follows:

$\overline{f}$ (mean),
σ_p^2 (variance),
$c_p = \sigma_p^2 e^{-\beta d}$ (covariance function),

where d – distance between two points 1 – (x_1, y_1) and 2 – (x_2, y_2) of the parameter space of the random field:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

β – correlation decay coefficient.

Local averages are rectangular, each has the length l and height t. According to [6], the local average is a random variable that can be expressed in terms of the stochastic integral:

$$f^m = \frac{1}{lt} \int_0^l \int_0^t f(x,y)\, dx dy. \tag{3.1}$$

The isotropy of the initial random field guarantees that the mean remains unchanged

$$E\left[f^m\right] = \frac{1}{lt} \int_0^l \int_0^t E\left[f(x,y)\right] dx dy = \overline{f}.$$

The derivation of the local average variance is more tedious but not difficult:

$$\operatorname{Var}\left[f^m\right] = \sigma_u^2 = E\left[(f^m)^2\right] - \left(\overline{f}\right)^2. \tag{3.2}$$

When (3.1) is inserted into (3.2), then

$$\sigma_v^2 = \frac{1}{l^2 t^2} E\left[\left[\int_0^l \int_0^t f(x,y)\, dx dy\right]^2\right] - \left(\overline{f}\right)^2.$$

It can be proved that

$$E\left[\left[\int_0^l \int_0^t f(x,y)\, dx dy\right]^2\right] = \int_0^l \int_0^t \int_0^l \int_0^t E[f(x,y) f(z,u)]\, dx dy du dz \tag{3.3}$$

and

$$E[f(x,y) f(z,u)] = \operatorname{cov}\left[f(x,y), f(z,u)\right] + \left(\overline{f}\right)^2. \tag{3.4}$$

Let us now choose two points $P(x,y)$ and $Q(x+u,y+z)$, pertaining to the element, such that $u,z \geq 0$. Their correlation can be expressed as

$$g(u,z) = e^{-\beta\sqrt{u^2+z^2}}.$$

By inserting (3.4) into (3.3) and using (3.2), we obtain the following expression to calculate the local average variance:

$$(3.5)\qquad \sigma_u^2 = \frac{\sigma_p^2}{l^2t^2}\int_0^l\int_0^t\left(\int_0^x\int_0^y g\,du\,dz + \int_0^x\int_0^{t-y} g\,du\,dz + \int_0^{l-x}\int_0^y g\,du\,dz + \int_0^{l-x}\int_0^{t-y} g\,du\,dz\right)dx\,dy,$$

hence

$$(3.6)\qquad \sigma_u^2 = \frac{\sigma_p^2}{l^2t^2}\int_0^l\int_0^t (I_1(x,y)+I_2(x,y)+I_3(x,y)+I_4(x,y))\,dx\,dy.$$

The sum in the integral in (3.6) expresses the correlation of two random variables – first at a point $P(x,y)$ and the second one – averaged over the whole finite element. Formulas (3.5) and (3.6) can be presented in the closed form provided that the function $g(u,z)$ is analytically integrable. Unfortunately this is not the case here and only numerical integration is possible.

The derivation of formulas for covariances is quite similar. If the elements (local averages in terms of random fields) do not partly overlap each other, two cases can be distinguished (Fig. 1):

a. Projections of both elements on one axis of the global coordinate system fully coincide; if the distance of their centroids is equal to ξ, we may write (3.5) as

$$(3.7)\qquad \operatorname{cov}[\mathrm{I},\mathrm{II}] = c_u = \frac{\sigma_p^2}{l^2t^2}\int_0^l\int_0^t\left(\int_0^y\int_{\xi-x}^{\xi+l-x} g\,du\,dz + \int_0^{t-y}\int_{\xi-x}^{\xi+l-x} g\,du\,dz\right)dx\,dy.$$

Hence (3.6) transforms into

$$(3.8)\qquad c_u = \frac{\sigma_p^2}{l^2t^2}\int_0^l\int_0^t (I_1(x,y)+I_2(x,y))\,dx\,dy.$$

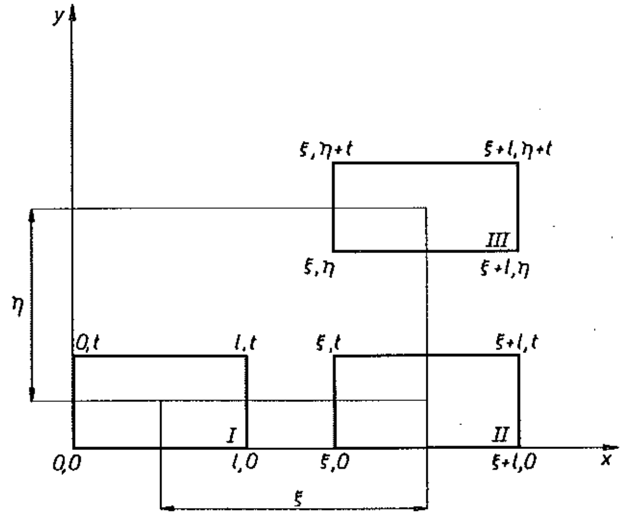


FIG. 1. Rectangular local averages I, II and III.

b. Projections do not coincide at all, so there are two components of the distance between the centroids – horizontal ξ and vertical η; the corresponding expressions take forms of (3.9) and (3.10), respectively:

$$\text{cov}\,[\text{I},\text{III}] = c_u = \frac{\sigma_p^2}{l^2t^2}\int\limits_0^l\int\limits_0^t\int\limits_{\xi-x}^{\xi+l-x}\int\limits_{\eta-y}^{\eta+t-y} g\,du\,dz\,dx\,dy, \tag{3.9}$$

$$c_u = \frac{\sigma_p^2}{l^2t^2}\int\limits_0^l\int\limits_0^t I_1(x,y)\,dx\,dy. \tag{3.10}$$

In case the elements partly overlap each other ($\xi < l$ and $\eta < t$), the formula can be derived in a similar manner:

$$\text{cov}\,[\text{I},\text{III}] = \frac{\sigma_p^2}{l^2t^2}\int\limits_0^l\int\limits_0^t \sum_{i=1}^{9} I_i(x,y)\,dx\,dy. \tag{3.11}$$

In case $\xi < l$ and $\eta = 0$ the formula (3.11) takes the form

$$\text{cov}\,[\text{I},\text{II}] = \frac{\sigma_p^2}{l^2t^2}\int\limits_0^l\int\limits_0^t \sum_{i=4}^{9} I_i(x,y)\,dx\,dy, \tag{3.12}$$

where

$$I_1 = \int\limits_{\xi-x}^{\xi+l-x} \int\limits_{\eta-y}^{\eta+t-y} g\,du\,dz, \quad I_2 = \int\limits_{0}^{x} \int\limits_{\eta-y}^{\eta+t-y} g\,du\,dz, \quad I_3 = \int\limits_{0}^{l-x} \int\limits_{\eta-y}^{\eta+t-y} g\,du\,dz,$$

$$I_4 = \int\limits_{\xi-x}^{\xi+l-x} \int\limits_{0}^{y} g\,du\,dz, \quad I_5 = \int\limits_{\xi-x}^{\xi+l-x} \int\limits_{0}^{t-y} g\,du\,dz, \quad I_6 = \int\limits_{0}^{x} \int\limits_{0}^{y} g\,du\,dz,$$

$$I_7 = \int\limits_{0}^{l-x} \int\limits_{0}^{y} g\,du\,dz, \quad I_8 = \int\limits_{0}^{x} \int\limits_{0}^{t-y} g\,du\,dz, \quad I_9 = \int\limits_{0}^{l-x} \int\limits_{0}^{t-y} g\,du\,dz.$$

The value I_1 in (3.10) and the sum $I_1 + I_2$ in (3.8) and the sums $\sum_i I_i$ in (3.11) and (3.12) represent the correlation of the field at a point $P(x,y)$ pertaining to element I and the local average II or III. Like in case of the local average variance, only numerical integration is possible.

4. Numerical results

To examine the anisotropy, local averages must be arranged in such a manner that the distances of their centroids are constant (Fig. 2). Thus the following holds true:

$$z = \sqrt{\xi^2 + \eta^2} = \text{const}.$$

For a given z we can express the anisotropy as a function of the angle ϕ only. The values of z are the multiplies of the element's diagonal ($1d$, $2d$, $3d$)

$$d = \sqrt{l^2 + t^2}.$$

The angle ϕ is such that

$$\tan\phi = \frac{\eta}{\xi}.$$

It equals 0°, 15°, 30°, 45°, 60°, 75° and 90°. All rectangles have the same height $t = 1$ and their lengths are equal to 1, 1.25, 1.5, 1.75, 2 and 2.5. The analysis covers a wide range of correlation parameters such that the product βl equals 0.1, 0.5, 1.0, 2.0 and 5.0. For all calculations the constant value of $\sigma_p^2 = 1$ was assumed.

In Fig. 3 (Table 1) the variances σ_u^2 and in Figs. 4a to 4f the ratios c_u/c_p are drawn. Since $\sigma_p^2 = 1$, the results for σ_u^2 can also be treated as ratios of

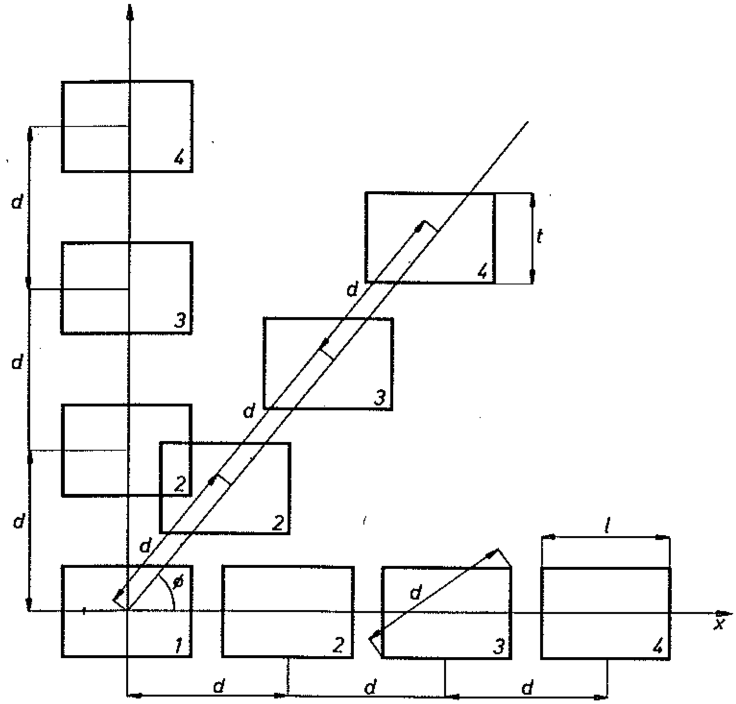


FIG. 2. Local averages in anisotropy investigation. $l = 1$, 1.25, 1.5, 1.75, 2 and 2.5; $t = 1$ for all cases; $\phi = 0°$, 15°, 30°, 45°, 60°, 75°, 90°.

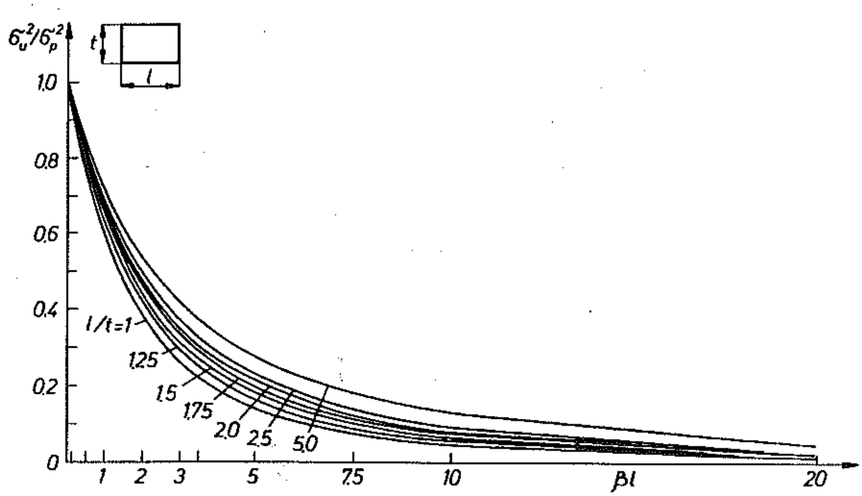


FIG. 3. σ_u^2/σ_p^2 ratios v.s. βl.

σ_u^2/σ_p^2. The values of c_p do not depend on direction, so the Figs. 4a to 4f are a good illustration of angular anisotropy.

Table 1.

$l/t\backslash\beta l$	.1	0.5	1.0	2.0	3.5	5.0	7.5	10	20
1.0	0.9483	0.7717	0.6049	0.3885	0.2209	0.1384	0.0747	0.0462	0.0136
1.25	0.9531	0.7909	0.6339	0.4226	0.2500	0.1609	0.0889	0.0537	0.0165
1.50	0.9562	0.8035	0.6534	0.4471	0.2729	0.1796	0.1015	0.0645	0.0194
1.75	0.9583	0.8122	0.6675	0.4655	0.2910	0.1951	0.1127	0.0725	0.0222
2.0	0.9598	0.8186	0.6777	0.4796	0.3056	0.2081	0.1225	0.0798	0.02497
2.5	0.9618	0.8269	0.6918	0.4995	0.3274	0.2285	0.1389	0.0930	0.0300
5.0	0.9650	0.8414	0.7168	0.5376	0.3735	0.2758	0.1824	0.1301	0.0500

In Tables 2, 3 and 4 more thorough results are presented. They exhibit the greatest relative differences between covariances of local averages for the same distance z:

$$e_p = \frac{\max(c_u) - \min(c_u)}{\max(c_u)} 100\% .$$

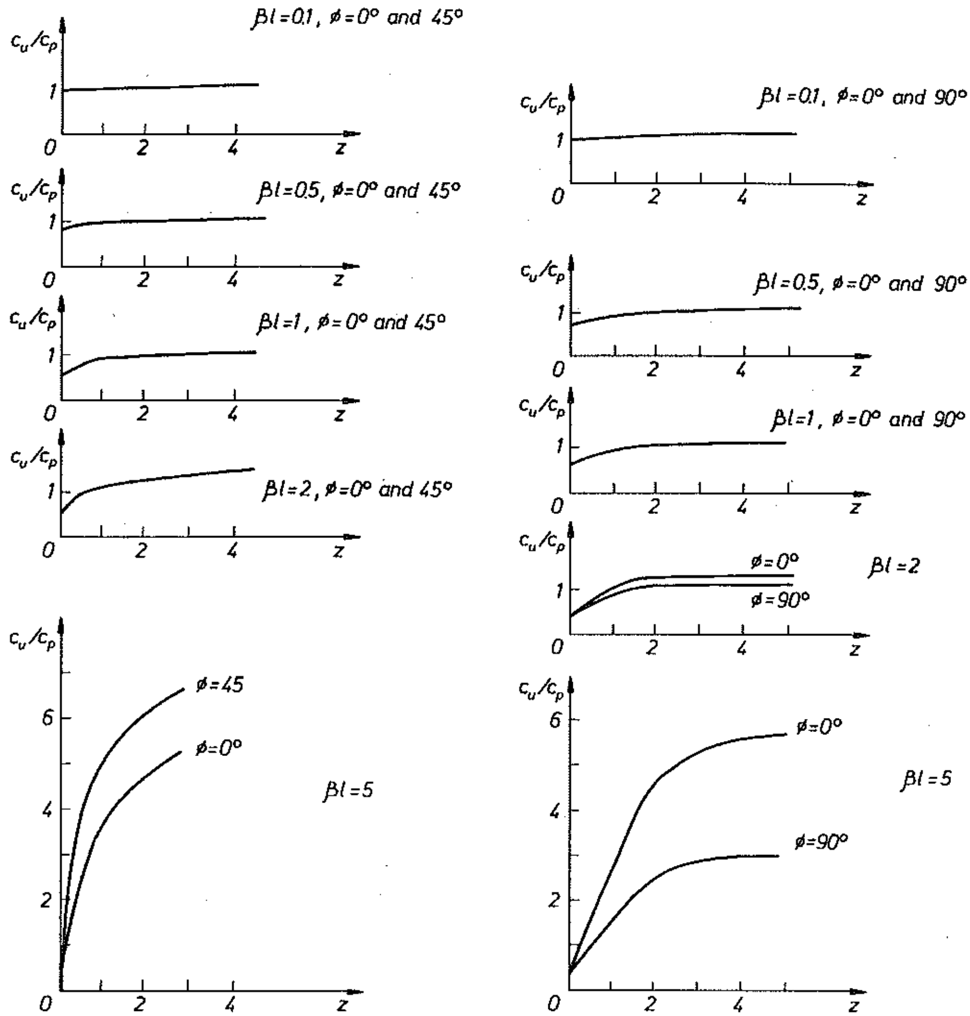
In Table 2 the results for neighbouring elements ($z = d$) are presented, in Table 3 for medium-distant ones ($z = 2d$), and in Table 4 for the most remote one ($z = 3d$).

Table 2.

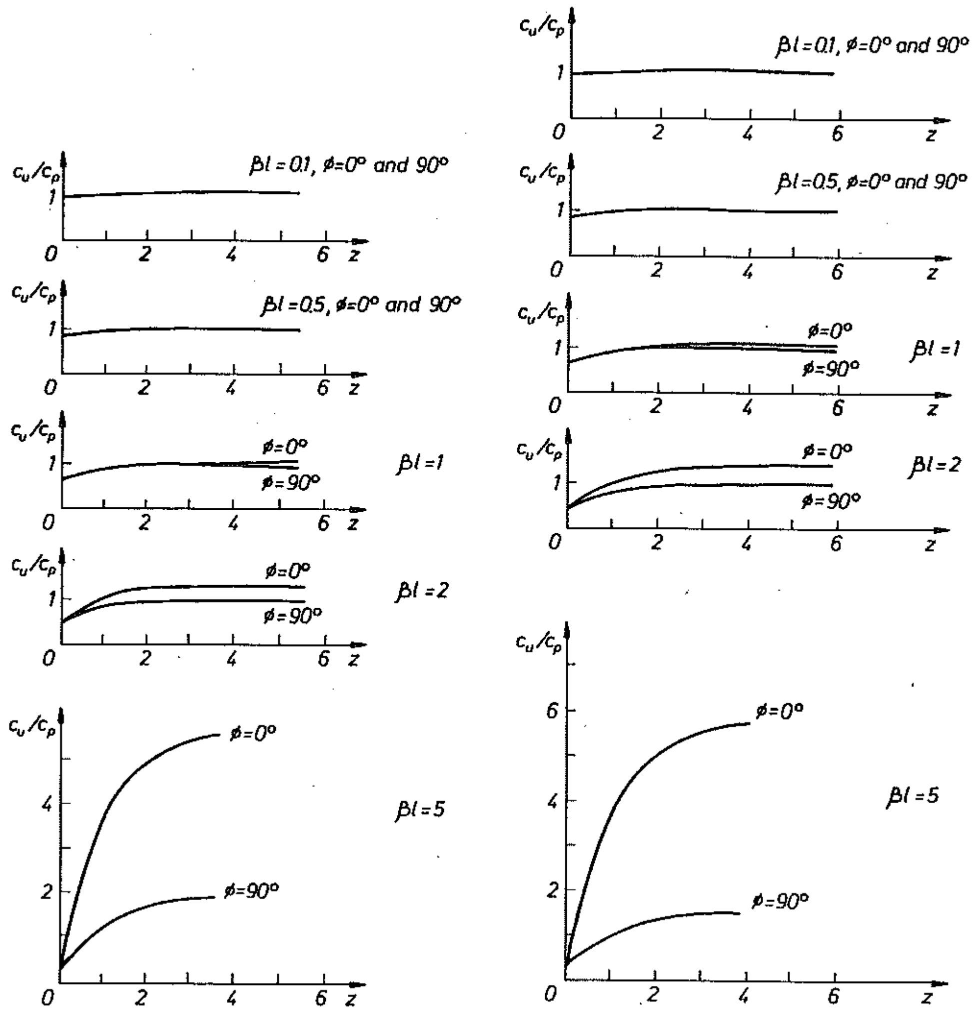
$l/t\backslash\beta l$	.1	0.5	1.0	2.0	5.0
1.0	0%	0%	0.5%	2.6%	25% v.s.n.
1.25	0%	1.8%	5.0%	21%	46% v.s.n.
1.50	0%	2.8%	7.6%	21%	64% v.s.n.
1.75	0.5%	3.5%	9.4%	25%	72% v.s.n.
2.00	0.6%	4.1%	10.5%	28%	77% v.s.n.
2.50	0.7%	4.6%	12.0%	32%	81% v.s.n.

Table 3.

$l/t\backslash\beta l$	.1	0.5	1.0	2.0	5.0
1.0	0%	0%	1.6%	2.1%	19% v.s.n.
1.25	0%	1.4%	4.3%	21%	not computed
1.50	0%	2.2%	6.4%	19%	64% v.s.n.
1.75	0.5%	2.7%	8.0%	24%	64% v.s.n.
2.00	0.6%	3.0%	9.0%	26%	76% v.s.n.
2.50	0.7%	3.0%	10.0%	29%	90% v.s.n.


βl=0.1, ϕ=0° and 45°
βl=0.5, ϕ=0° and 45°
βl=1, ϕ=0° and 45°
βl=2, ϕ=0° and 45°
ϕ=45
ϕ=0°
βl=5
βl=0.1, ϕ=0° and 90°
βl=0.5, ϕ=0° and 90°
βl=1, ϕ=0° and 90°
ϕ=0°
ϕ=90°
βl=2
ϕ=0°
ϕ=90°
βl=5
c_u/c_p
z

[Fig. 4a,b]



βl = 0.1, ϕ=0° and 90°
βl = 0.5, ϕ=0° and 90°
ϕ=0°
ϕ=90°
βl = 1
βl = 2
βl = 5
c_u/c_p
z
βl = 0.1, ϕ=0° and 90°
βl = 0.5, ϕ=0° and 90°
ϕ=0°
ϕ=90°
βl = 1
βl = 2
βl = 5
c_u/c_p
z

[FIG. 4c,d]

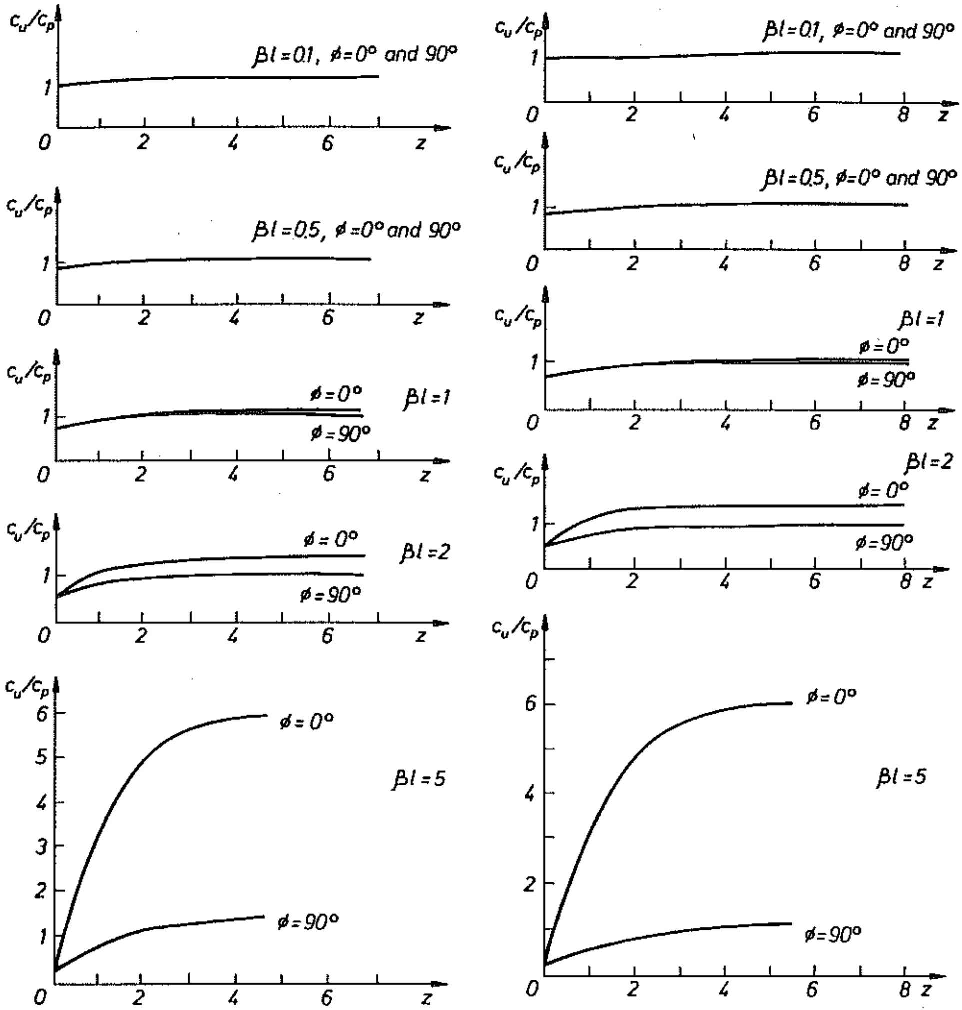


FIG. 4. c_u/c_p ratios vs. distance between centroids z for: a) $l/t = 1$, b) $l/t = 1.25$, c) $l/t = 1.5$, d) $l/t = 1.75$, e) $l/t = 2$, f) $l/t = 2.5$.

Table 4.

$l/t \backslash \beta l$	.1	0.5	1.0	2.0	5.0
1.0	0%	0%	0.6%	not computed – v.s.n.	not computed – v.s.n.
1.25	0%	1.2%	4.0%	14% v.s.n.	47% v.s.n.
1.50	0%	1.8%	5.8%	20% v.s.n.	not computed – v.s.n.
1.75	0%	2.3%	7.0%	21% v.s.n.	not computed – v.s.n.
2.0	0%	2.6%	8.2%	23% v.s.n.	not computed – v.s.n.
2.5	0%	2.9%	9.0%	27% v.s.n.	not computed – v.s.n.

v.s.n. – very small numbers

5. Conclusions

Discretization of the parameter space of any isotropic random field with finite elements always causes the anisotropy of the equivalent finite random variable of local averages. This is also true for squares, but their anisotropy is practically negligible (the curves in Fig. 4a do not bifurcate for $\beta l \leq 2$).

Slightly flattened rectangles ($l/t = 1.25$) should satisfy a stronger constraint to minimize the resulting anisotropy such that $\beta l \leq 1$ (see Fig. 4b).

Flattened rectangles should obey a tougher limitation $\beta l \leq 0.5$ (Figs. 4c to f).

The values of covariances decay with the increase of the distance z between the centroids of the elements. The ratio for the extreme ones may be great, but the anisotropy for large z is meaningless since both covariances are then close to zero. This conclusion is important for non-homogeneous meshes often used in practice. The anisotropy matters only in the vicinity of loads and only there it should be minimized.

These constraints should be taken into account in engineering practice with additional guidelines:

- the stochastic mesh should exactly match the deterministically optimum mesh in the vicinity of loaded nodes to obtain accurate simulation results;
- the elements close to loaded nodes should be square to guarantee minimum anisotropy; more remote elements practically do not affect the results due to very small covariances;
- small covariances of distant elements indicate that deterministic elements distant from loaded nodes can be combined into larger local averages to simplify the simulation, without adverse effects on the accuracy of results;

• it appears to be not difficult to adapt the existing deterministic codes to MC simulations in which the deterministic mesh is equal to the stochastic one; a routine containing the calculation of $\sigma^2 u$, cov [I, II] and cov [I, III] should be added together with random field generation and statistical analysis;

• it seems to be difficult to adapt the existing deterministic codes to simulations in which deterministic mesh is not equal to the stochastic one, because the stochastic mesh must be generated depending on both the load distribution and the deterministically optimum mesh in a given application.

Variances σ_u^2 can be expressed in terms of two dimensionless parameters βl and l/t.

Similarly, the covariances c_u can be expressed in terms of four dimensionless parameters βl, βt, $\beta \xi$ and $\beta \eta$ or βl, l/t, $\beta \xi$ and $\beta \eta$.

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