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ROBUST STABILITY OF DYNAMICAL SYSTEMS

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A general concept of the robust stability of uncertain nonlinear dynamical systems is given. By using the method of optimal Lyapunov functions, the robust stability analysis is performed in the general case of a multidimensional system described by ordinary differential equations. The presented approach is applied to the problem of stability of affine systems with nonstationary structural disturbances. An illustrative example of a perturbed oscillator is given.

1. Introduction

There are many methods of stability analysis of nonautonomous dynamical systems described by differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \qquad \mathbf{x} \in R^n, \tag{1.1}$$

where $\mathbf{f}(\mathbf{0}, t) = \mathbf{0}$ for every $t \geq t_0$. The approach applied to the problem, and particularly the applied definition of stability is usually dependent on the concrete form of the function $\mathbf{f}$ as well as on the aim of the analysis. In science and technology we very often deal with problems of stability of uncertain dynamical systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}), \qquad \mathbf{x} \in R^n, \qquad \mathbf{f}(\mathbf{0}, \mathbf{p}) = \mathbf{0}, \tag{1.2}$$

where parameters $\mathbf{p} \in R^k$ are not known exactly or can vary in time. That is why the above problem of stability (the so-called robust stability) has received much attention in scientific literature (e.g. [1 – 8]) for several years.

The robust stability analysis is usually based on the method of Lyapunov functions and many interesting results have been obtained mainly for linear uncertain systems (e.g. [1, 2, 4, 5, 6, 8]). Relatively small number of works concern essentially nonlinear systems (e.g. [3, 7, 9, 14]). Moreover, as it can be concluded from the literature, there are various approaches to the robustness as a stability property of dynamical systems. Therefore, optimal estimates of the stability region of a given system can be defined in different ways what may lead to some misunderstandings.

The main purpose of this work is to provide not a completely new method of robust stability analysis of uncertain dynamical systems but rather a general concept of the robust stability, bringing together the most popular approaches to robustness that are found in the literature. The presented concept of robust stability analysis is based on the method of optimal Lyapunov functions ([3, 9, 11]).

In Sec. 2 a general concept of the robust stability of uncertain dynamical systems is described in terms of families of dynamical systems. Using the method of optimal Lyapunov functions, various optimization problems are formulated in Sec. 3. In particular, since the results of robust stability analysis obtained by a unique Lyapunov function are usually conservative, it is shown in Sec. 3 how to improve stability estimates for a given uncertain system by means of many optimal Lyapunov functions ([5]). The presented approach is applied in Sec. 4 to the problem of robust stability of a wide class of nonlinear uncertain systems. In particular, a class of multidimensional linear systems under nonstationary perturbations is considered in details. An illustrative example of a disturbed linear oscillator is given in Sec. 5. Some useful formulae and propositions concerning linear systems are provided in the Appendix.

2. General concept of robust stability

If we consider an uncertain system (1.2) with unknown parameters $\mathbf{p}$ belonging to a subset $P \subset R^k$ for every $t > t_0$, then we deal, in fact, with the following family of dynamical systems

$$(2.1) \qquad (\mathbf{f}, P) := \left\{ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}) : \quad \mathbf{x} \in R^n, \ \ \mathbf{p} \in P \subset R^k \right\}.$$

Therefore, we will identify an uncertain system with the corresponding family of dynamical systems.

Since we will be interested, in this paper, in the global asymptotical stability of the stationary point $\mathbf{x} = \mathbf{0}$ of an uncertain system (2.1) in the state space R^n, we assume that the origin is the unique stationary point of (1.2) for each $\mathbf{p} \in P$. Moreover, we assume that $\mathbf{f}$ is such a function of $\mathbf{x}$, $\mathbf{p}$ that sufficient conditions for the existence of solutions of equation (1.2) are satisfied.

In many practical problems of stability we can distinguish a constant vector $\mathbf{p}_0 \in P$ of nominal values of parameters of the system for which the system is stable. Then real parameters $\mathbf{p}$ of the system can be decomposed as follows

$$\mathbf{p} = \mathbf{p}(t) = \mathbf{p}_0 + \mathbf{z}(t), \tag{2.2}$$

where $\mathbf{z}(t)$ is a vector of perturbations which can be in general nonstationary. Thus any dynamical system with perturbations can be represented by an uncertain system (2.1) with varying parameters.

Since in real systems the possible variations of parameters are always bounded we assume, without any essential loss of generality, that P is a compact set in R^k although such an assumption will not be always necessary for our further considerations. In other words, we assume that the set Z of admissible values of the perturbations is compact in R^k.

It is natural to introduce the following definition

DEFINITION 1. *An uncertain system* $(\mathbf{f}, P)$ *is stable in* P *if and only if for each fixed* $\mathbf{p} \in P$ *the system* (1.2) *is stable as an autonomous dynamical system.*

In the above sense the robustness of uncertain systems is understood in some papers (e.g. [1, 2, 3]). It is clear, that this kind of stability (we say: weak robust stability – WRS) is of practical meaning in such cases only when parameters of a given system are constant in time, although they can be perturbed. However, in many practical problems parameters of the system can vary in time, for example due to external nonstationary perturbations. Then, it is necessary to consider a stronger stability property, namely, strong robust stability – SRS.

DEFINITION 2. *An uncertain system* $(\mathbf{f}, P)$ *is strongly robustly stable in* P *if and only if for every function* $\mathbf{p} : \langle t_0, \infty) \to P$ *the system*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}(t)), \qquad t \geq t_0 \tag{2.3}$$

is stable as a nonautonomous dynamical system.

As it is seen, the weak robust stability ensures the stability of a given system (1.2) for each fixed vector of parameters $\mathbf{p}$ taken from $P \subset R^k$, contrary to the strong robust stability which ensures that the system remains stable when its parameters $\mathbf{p}$ are varying in time within the bounds determined by P. However, in some cases we want only the system to be stable locally in P, i.e. to be stable under sufficiently small nonstationary perturbations of parameters in a neighbourhood of each point $\mathbf{p} \in P$, then it is sufficient to study the so-called locally strong robust stability – LSRS.

DEFINITION 3. *An uncertain system* $(\mathbf{f}, P)$ *is locally, strongly robustly stable if and only if for each* $\mathbf{p}_0 \in P$ *there exists an open, bounded neighbourhood*

$U_0 \subset R^k$, containing $\mathbf{p}_0$ and such that for every function $\mathbf{p} : \langle t_0, \infty) \to \overline{U}_0$ the system (2.3) is stable as a nonautonomous dynamical system (i.e. the uncertain system $(\mathbf{f}, \overline{U}_0)$ is strongly robustly stable).

The above three natural definitions of the robust stability of uncertain systems can be combined with any classical definition of stability of dynamical systems without uncertainties, and particularly with asymptotical stability. To specify our considerations, we use in this paper the following definitions of asymptotical stability which have a clear, practical meaning and are closely related to the second method of Lyapunov of stability investigation. Namely:

• *Exponential stability* (ES) (see e.g. [9, 13]).

DEFINITION 4. *A dynamical system* (1.1) *is globally exponentially stable if there exist positive real numbers λ, γ such that the inequality*

$$\|\mathbf{x}(t)\| \le \lambda \|\mathbf{x}(t_0)\| e^{-\gamma(t-t_0)} \tag{2.4}$$

is satisfied for every $\mathbf{x}_0 \in R^n$ and $t \ge t_0$.

• *Quadratic stability* (QS) (see [4]).

DEFINITION 5. *A dynamical system* (1.1) *is quadratically asymptotically stable if there exists a positive definite symmetric matrix $\mathbf{S}$ and a scalar $\lambda > 0$ such that*

$$\mathbf{x}^T \mathbf{S}\, \mathbf{f}(\mathbf{x}, t) + \mathbf{f}^T(\mathbf{x}, t) \mathbf{S}\, \mathbf{x} \le -\lambda \|\mathbf{x}\|^2 \tag{2.5}$$

for every $\mathbf{x} \in R^n$, $t \ge t_o$ and for certain norm $\|\cdot\|$ in R^n.

• *Exponential stability with respect to a norm* (GS) (see [3, 9, 11]).

DEFINITION 6. *A dynamical system* (1.1) *is globally exponentially stable with respect to a norm $\|\cdot\|$ if there exists a real scalar $\gamma > 0$ such that*

$$\|\mathbf{x}(t)\| \le \|\mathbf{x}_0\| \exp[-\gamma(t - t_0)] \tag{2.6}$$

for every initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0 \in R^n$.

Applying the theorem on the equivalence of norms in R^n it is easy to deduce from Definitions 4, 5, 6 that

• the function $V_{\mathbf{S}}(\mathbf{x}) = \mathbf{x}^T \mathbf{S}\, \mathbf{x}$, with $\mathbf{S} > 0$ satisfying (2.5), is a Lyapunov function of the system,

• QS implies GS with respect to a certain norm $\|\cdot\|_{\mathbf{S}} = \sqrt{\mathbf{x}^T \mathbf{S}\, \mathbf{x}}$,

• GS implies ES,

• ES with a constant $\gamma > 0$ is a topological property of the system, i.e. it does not depend on the choice of norm.

For the above reasons we restrict, for convenience, our further considerations to the class of norms of the form $\|\cdot\|_{\mathbf{S}}$ (i.e. norms determined by a positive definite quadratic forms), although we do not always make use of this assumption.

Contrary to the exponential stability, the stabilities GS and QS are strongly dependent on the choice of norm. In particular, a given dynamical system stable exponentially with respect to a norm can be unstable exponentially with respect to another norm. However, as we will see later, these kinds of stabilities are more constructive and can provide in some cases more detailed information about the behaviour of trajectories of the system.

Any real number $\gamma > 0$ satisfying (2.4), (2.6) will be called the stability index for system (1.1). It is clear that the optimal index satisfying (2.4) (or (2.6)), i.e. the index

$$\gamma^* = \sup \gamma = \inf_{\mathbf{x}_0 \in R^n} \inf_{t \geq t_0} \frac{1}{(t - t_0)} \ln \frac{\|\mathbf{x}(t)\|_{\mathbf{S}}}{\eta \|\mathbf{x}_0\|_{\mathbf{S}}}, \tag{2.7}$$

where $\eta = \lambda > 0$ is fixed (or $\eta = 1$, respectively), is the most desirable since it is simply the exponential rate of convergence of trajectories of the system. Unfortunately, the stability index defined in such a way can rarely be determined because calculations performed according to (2.7) require the knowledge of solutions $\mathbf{x}(t)$ of the system for any initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. For this reason it is usually necessary to find a suitable estimate of the exponential rate of convergence. To do that we can apply condition (2.5) for quadratic stability. If there exists a positive definite matrix $\mathbf{S}$ satisfying (2.5) then, as one can easily deduce from the equivalence of norms in R^n, there also exists a positive constant γ such that $\dot{V}_{\mathbf{S}}(\mathbf{x}) \leq -2\gamma V_{\mathbf{S}}(\mathbf{x})$ on the trajectories of the system, i.e.

$$\forall_{\mathbf{x} \in R^n} \ \mathbf{x}^T \mathbf{S}\, \mathbf{f}(\mathbf{x}, t) + \mathbf{f}^T(\mathbf{x}, t) \mathbf{S}\, \mathbf{x} \leq -2\gamma \|\mathbf{x}\|_{\mathbf{S}}^2. \tag{2.8}$$

The above inequality leads directly to the exponential convergence (2.6). Since the constant γ in (2.8) satisfies the inequality

$$\gamma \leq -\mathbf{x}^T \mathbf{S}\, \mathbf{f}(\mathbf{x}, t) / (\mathbf{x}^T \mathbf{S}\, \mathbf{x})$$

for every $\mathbf{x} \neq \mathbf{0}$, an optimal estimate of the exponential rate of convergence can be calculated from the following formula (see e.g. [3, 11])

$$\widehat{\gamma} = \widehat{\gamma}(\mathbf{S}) = -\sup \frac{\mathbf{x}^T \mathbf{S}\, \mathbf{f}(\mathbf{x}, t)}{\mathbf{x}^T \mathbf{S}\, \mathbf{x}}, \tag{2.9}$$

where the supremum is over $\mathbf{x} \neq \mathbf{0}$, $t \geq t_0$. The major advantage of such a definition is that it does not assume the knowledge of solutions $\mathbf{x}(t)$ of the system and therefore it is applicable. For these reasons we apply in further considerations the following practical definition of exponential stability with respect to a norm.

DEFINITION 7. *A nonautonomous system* (1.1) *is said to be exponentially stable with respect to a norm* $\|\cdot\|_{\mathbf{S}}$ *if and only if the stability index calculated from formula* (2.9) *is positive.*

This kind of exponential stability will be denoted shortly by NS. Obviously, NS implicates GS with respect to the same norm.

In the particular case of linear stationary systems the equivalence QS⇔ES is true. In fact, as we know from the theory of linear systems, any linear stationary system $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x}$ is stable exponentially in R^n if and only if all eigenvalues of the matrix $\mathbf{A}$ have negative real parts, i.e. $\mathbf{A}$ is stable. Moreover, if the matrix $\mathbf{A}$ is stable, then for every positive definite matrix $\mathbf{Q}$ there exists a positive definite solution $\mathbf{S}$ of the so-called Lyapunov equation

$$\mathbf{A}^T\mathbf{S} + \mathbf{S}\,\mathbf{A} = -\mathbf{Q}\,. \tag{2.10}$$

Therefore, there exists a constant $\lambda > 0$ such that

$$\underset{\mathbf{x}\in R^n}{\forall}\ \mathbf{x}^T(\mathbf{A}^T\mathbf{S} + \mathbf{S}\,\mathbf{A})\mathbf{x} \le -\lambda\|\mathbf{x}\|^2_{\mathbf{S}}\,, \tag{2.11}$$

i.e. the system is quadratically stable or, more precisely, it is exponentially stable with respect to the norm $\|\cdot\|_{\mathbf{S}}$. Hence, any stationary linear system is exponentially stable if and only if it is quadratically stable (is exponentially stable with respect to a certain norm $\|\cdot\|_{\mathbf{S}}$).

Definition (2.9) can be directly adopted to any uncertain system $(\mathbf{f}, U)$, $U \subset P$. It is easy to see that the index for $(\mathbf{f}, U)$ should be calculated as follows

$$\widehat{\gamma}(\mathbf{S}) = -\sup_{\mathbf{x}\neq\mathbf{0}}\ \sup_{\mathbf{p}\in U} \frac{\mathbf{x}^T\mathbf{S}\,\mathbf{f}(\mathbf{x},\mathbf{p})}{\mathbf{x}^T\mathbf{S}\,\mathbf{x}}\,. \tag{2.12}$$

The above formula expresses the fact that the exponential rate of convergence for a given uncertain system is determind by the most disadvantageous critical perturbations of parameters, namely the perturbations $\widehat{\mathbf{z}}$ that maximize the function under supremum (2.12) at each point $\mathbf{x} \neq \mathbf{0}$ of the state space. Therefore, $\widehat{\mathbf{z}} = \widehat{\mathbf{z}}(\mathbf{x})$ i.e. the critical dynamics of the uncertain system is realized in a feedback system described by the following autonomous equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}_0 + \widehat{\mathbf{z}}(\mathbf{x})). \tag{2.13}$$

Thus, in order to estimate robust stability properties of the uncertain system $(\mathbf{f}, U)$ it is sufficient to study exponential stability of the corresponding equivalent autonomous feedback system (2.13).

It is clear that the index defined by (2.12) is in general dependent on the assumed set of parameters $U \subset P$ i.e. $\gamma = \gamma(\mathbf{S}, U)$. In the particular case $U = \{\mathbf{p}\}$ the index $\gamma = \gamma(\mathbf{S}, \mathbf{p})$.

Replacing the word "stability" in Definitions 1, 2, 3 (replacing the last letter "S" in symbols WRS, LSRS, SRS) by exponential stability (ES), quadratic stability (QS) or by exponential stability with respect to a norm (NS), one can produce directly the following kinds of robust stabilities: WRES, LSRES, SRES, WRQS, LSRQS, SRQS, LSRNS, SRNS, SRNS, respectively. The meaning of any letter in the above key words is obvious (for example, LSRQS should be read as *locally strong, robust, quadratic stability*).

The summary of the introduced definitions of robust stabilities of an uncertain system $(\mathbf{f}, P)$ is presented below:

$$\left.\begin{array}{ll}
\text{WRES}: & \forall_{\mathbf{p}\in P} \\
\text{LSRES}: & \forall_{\mathbf{p}_0\in P}\ \exists_{U_0}\ \forall_{\mathbf{p}:T\to\overline{U}_0} \\
\text{SRES}: & \forall_{\mathbf{p}:T\to P}
\end{array}\right\} \left\{\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{p}) \text{ is exponentially stable}\right\},$$

$$\left.\begin{array}{ll}
\text{WRNS}: & \exists_{\mathbf{S}>0}\ \forall_{\mathbf{p}\in P} \\
\text{WRQS}: & \forall_{\mathbf{p}\in P}\ \exists_{\mathbf{S}>0} \\
\text{SRNS}: & \exists_{\mathbf{S}>0}\ \forall_{\mathbf{p}:T\to P} \\
\text{SRQS}: & \forall_{\mathbf{p}:T\to P}\ \exists_{\mathbf{S}>0} \\
\text{LSRNS}: & \exists_{\mathbf{S}>0}\ \forall_{\mathbf{p}_0\in P}\ \exists_{U_0}\ \forall_{\mathbf{p}:T\to\overline{U}_0} \\
\text{LSRQS}: & \forall_{\mathbf{p}_0\in P}\ \exists_{\mathbf{S}>0}\ \exists_{U_0}\ \forall_{\mathbf{p}:T\to\overline{U}_0}
\end{array}\right\} \left\{\begin{array}{c}\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{p}) \text{ is exponentially stable} \\ \text{with respect to the norm } \|\cdot\|_{\mathbf{S}}\end{array}\right\},$$

where $T = \langle t_0, \infty)$ and U_0 is an open neighbourhood of $\mathbf{p}_0$.

We assume in our further considerations that the stability index $\widehat{\gamma}(\mathbf{S}, \mathbf{p})$ is a continuous function of parameters $\mathbf{p}$ (in Sec. 4 we prove this continuity property for a wide class of uncertain systems). We also assume that the set of admissible parameters P is compact in R^k. Then it is easy to see that the introduced robust stabilities satisfy certain logical relations which are illustrated in Fig. 1.

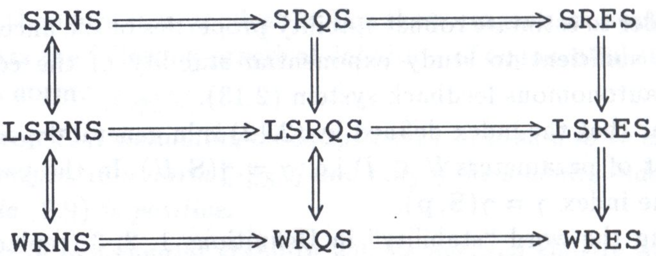

FIG. 1. The diagram of logical relations between various kinds of robust stabilities.

Most of the above relations are obvious. Therefore, we prove only the implications: (WRNS $\Rightarrow$ LSRNS $\Rightarrow$ SRES) and (WRQS $\Rightarrow$ LSRQS).

THEOREM 1. WRNS $\Rightarrow$ LSRNS $\Rightarrow$ SRNS.

P r o o f. Suppose that an uncertain system $(\mathbf{f}, P)$ is weakly robustly exponentially stable with respect to a norm. Then, there exists a positive definite $n \times n$ matrix $\mathbf{S}$ such that for every $\mathbf{p} \in P$ the index $\widehat{\gamma}(\mathbf{S}, \mathbf{p})$ is positive. Therefore, since $\widehat{\gamma}(\mathbf{S}, \mathbf{p})$ is a continuous function of $\mathbf{p}$ and P is a compact set, also $\gamma = \inf\limits_{\mathbf{p} \in P} \widehat{\gamma}(\mathbf{S}, \mathbf{p})$ exists and is positive. But as it follows from (2.12)

$$\boldsymbol{\gamma} = \inf_{\mathbf{p} \in P} \left[-\sup_{\mathbf{x} \neq 0} \frac{\mathbf{x}^T \mathbf{S}\, \mathbf{f}(\mathbf{x})}{\mathbf{x}^T \mathbf{S}\, \mathbf{x}} \right] = -\sup_{\mathbf{x}, \mathbf{p}} \left[\frac{\mathbf{x}^T \mathbf{S}\, \mathbf{f}(\mathbf{x})}{\mathbf{x}^T \mathbf{S}\, \mathbf{x}} \right] = \widehat{\gamma}(\mathbf{S}, P) > 0,$$

i.e. the exponential rate of convergence $\widehat{\gamma}(\mathbf{S})$ of trajectories of the uncertain system is also positive. Thus $(\mathbf{f}, P)$ is strongly robustly exponentially stable with respect to the norm $\|\cdot\|_{\mathbf{S}}$.

THEOREM 2. WRQS $\Rightarrow$ LSRQS.

P r o o f. Let $P = \{\mathbf{p}_\sigma : \ \sigma \in \Sigma\}$. If an uncertain system $(\mathbf{f}, P)$ is weakly robustly quadratically stable, then for every $\mathbf{p}_\sigma \in P$ there exists a positive definite symmetric $n \times n$ matrix $\mathbf{S}_\sigma$ such that $\widehat{\gamma}(\mathbf{S}_\sigma, \mathbf{p}_\sigma) > 0$. Since $\widehat{\gamma}(\mathbf{S}_\sigma, \mathbf{p})$ is a continuous function of $\mathbf{p}$, it is also positive for $\mathbf{p}$ belonging to an open neighbourhood U_σ of $\mathbf{p}_\sigma$. Thus, the uncertain system $(\mathbf{f}, P)$ is also locally weakly robustly quadratically stable.

Furthermore, the following theorems are also valid in the case when the set P is compact.

THEOREM 3. *If an uncertain system $(\mathbf{f}, P)$ with a compact set of admissible values of parameters $P \subset R^k$ is weakly robustly quadratically stable then there exists a finite collection of norms $\|\cdot\|_i = \|\cdot\|_{\mathbf{S}_i}$, $i = 1, 2, \ldots, l$, such that for every $\mathbf{p} \in P$, there exists an open neighbourhood $U_{\mathbf{p}}$ of $\mathbf{p}$ such that the system $(\mathbf{f}, \overline{U}_{\mathbf{p}})$ is strongly robustly exponentially stable with respect to a norm from the collection.*

P r o o f. Since $P = \{\mathbf{p}_\sigma : \sigma \in \Sigma\}$, the collection of the corresponding open sets U_σ, $\sigma \in \Sigma$ forms an open cover of P i.e. $P \subset \bigcup_{\sigma\in\Sigma} U_\sigma$. Therefore, since P is a compact set, there exists a finite subcover $U_1 \cup \ldots \cup U_l$ of P. Hence, a finite collection of norms $\|\cdot\|_1, \ldots, \|\cdot\|_l$ is sufficient to describe the property of LSRQS of the system $(\mathbf{f}, P)$. This concludes the proof.

THEOREM 4 (see e.g. [13]). *If the index $\widehat{\gamma}(\mathbf{S}, \mathbf{p})$ of the system (1.2) is a continuous function of parameters $\mathbf{p} \in P$ and the system with nominal values of parameters $\mathbf{p}_0$ is exponentially stable with respect to the norm $\|\cdot\|_{\mathbf{S}}$, then the system maintains the stability for sufficiently small perturbations of nominal parameters.*

P r o o f. It follows from the assumption of stability of the nominal system that $\widehat{\gamma}(\mathbf{S}, \mathbf{p}_0) > 0$. Therefore, since $\widehat{\gamma}(\mathbf{S}, \mathbf{p})$ is a continuous function of $\mathbf{p}$, the index $\gamma(\mathbf{S}, \mathbf{p})$ is positive for any $\mathbf{p}$ belonging to an open neighbourhood U_0 of $\mathbf{p}_0$. Thus the system $(\mathbf{f}, \overline{U}_0)$ is strongly robustly exponentially stable with respect to the norm $\|\cdot\|_{\mathbf{S}}$.

3. OPTIMAL ROBUST STABILITY ANALYSIS

In problems of robust stability we usually look for best estimates of the stability properties of an uncertain system $(\mathbf{f}, P)$. To achieve this aim, an optimal robust stability analysis should be performed on the basis of suitable quality factors. According to our knowledge of the set P of admissible values of parameters of the system, three main, practical cases of such an analysis can be recognized:

1. P is known and completely determined

In such case all what we have to do is to study the robust stability of the system in P. If we are interested in WRNS (SRNS) with respect to a given norm $\|\cdot\|_{\mathbf{S}}$, we simply have to calculate a stability index $\gamma(\mathbf{S}, P) \le \widehat{\gamma}(\mathbf{S}, P)$ and check whether $\gamma(\mathbf{S}, P) > 0$.

In the other case, if the norm is not fixed (i.e. $\mathbf{S}$ belongs to a class $\mathbb{S}$ of positive definite $n \times n$ matrices), then we can consider the following problem of Lyapunov function optimization

$$\sup_{\mathbf{S}\in\mathbb{S}} \gamma(\mathbf{S}, P). \tag{3.1}$$

If there exists an optimal positive-definite matrix $\widehat{\mathbf{S}} \in \mathbb{S}$ (i.e. an optimal Lyapunov function $\mathbf{x}^T\widehat{\mathbf{S}}\,\mathbf{x}$) for which $\gamma(\widehat{\mathbf{S}}, P) > 0$, then the system is weakly (strongly) robustly exponentially stable with respect to the norm $\|\cdot\|_{\widehat{\mathbf{S}}}$, with the maximal value of the stability index $\gamma(\widehat{\mathbf{S}}, P)$.

If for every $\mathbf{S} > 0$ the index $\gamma(\mathbf{S}, P)$ is not positive, then we may hope that the system is at least robustly quadratically stable. In order to prove WRQS or LWRES it is sufficient to find a finite, open cover $U_1, U_2, \ldots, U_l$ of P and the corresponding Lyapunov matrices $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_l$ such that each system $(\mathbf{f}, \overline{U}_i)$, $i = 1, 2, \ldots, l$, is strongly robustly exponentially stable with respect to the corresponding norm $\|\cdot\|_i$.

2. The shape of P is determined, the size is unknown

In this case we look for an optimal set of the robust stability of an uncertain system in a given shape class of sets

$$\mathbb{P} = \left\{ P(\boldsymbol{\alpha}) \subset R^k : \ \boldsymbol{\alpha} \in A \subset R^l \right\},$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_l)$ is a vector of size parameters. We assume, for convenience, that $\alpha_i \geq 0$, $i = 1, 2, \ldots, l$; (i.e. $\boldsymbol{\alpha} \in R^l_+$) and $P(\mathbf{0})$ contains only one point $\mathbf{p}_0$ such that the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}_0)$, without uncertainties, is stable asymptotically. In such cases we shall use the simplified notation $\gamma(\mathbf{S}, \boldsymbol{\alpha})$ instead of $\gamma(\mathbf{S}, P(\boldsymbol{\alpha}))$.

The main aim of the optimal robust stability analysis is to find an optimal size vector $\widehat{\alpha} \in R^l$ such that the uncertain system is robustly stable in $P(\boldsymbol{\alpha})$. To do this, we have to introduce a suitable, real measure μ of the stability set $P(\boldsymbol{\alpha})$ as a quality factor for optimization. We can use, for example, one of the measures

$$\mu_i(P(\boldsymbol{\alpha})) = \alpha_i\,, \quad i = 1, 2 \ldots, l; \qquad \mu_0(P(\boldsymbol{\alpha})) := \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_l\,,$$

having a clear natural meaning, or another measures more adequate for our special purposes.

Suppose that the norm $\|\cdot\|_{\mathbf{S}}$ with respect to which the robust stability analysis is to be performed is fixed. Then, applying a suitable measure μ we can pose the following problem of optimization:

$$\sup_{\boldsymbol{\alpha} \in Q(\mathbf{S})} \mu(P(\boldsymbol{\alpha})), \tag{3.2}$$

where $Q(\mathbf{S}) := \{\boldsymbol{\alpha} \in R^l_+ : \ \gamma(\mathbf{S}, \boldsymbol{\alpha}) > 0\}$. In the result we obtain an optimal vector of size parameters $\widehat{\boldsymbol{\alpha}}$ and the corresponding stability set $P(\widehat{\boldsymbol{\alpha}})$, optimal in the shape class $P(\boldsymbol{\alpha})$.

It is clear that the optimal vector $\widehat{\boldsymbol{\alpha}}$, obtained in this way, depends, in general, on $\mathbf{S}$ i.e. $\widehat{\boldsymbol{\alpha}} = \widehat{\boldsymbol{\alpha}}(\mathbf{S})$. Therefore, if $\mathbf{S}$ is not fixed but belongs to a class

$\mathbb{S}$ of positive definite matrices, then we can perform further optimization (Lyapunov function optimization)

$$\sup_{\mathbf{S}\in\mathbb{S}} \mu\,[P(\hat{\boldsymbol{\alpha}}(\mathbf{S}))] \tag{3.3}$$

in order to find an optimal matrix $\widehat{\mathbf{S}}$, the corresponding optimal Lyapunov function and the estimate of the stability set $\widehat{P} = P[\hat{\boldsymbol{\alpha}}(\widehat{\mathbf{S}})]$ of the maximal measure μ in the shape class $P(\boldsymbol{\alpha})$.

Furthermore, if we are interested in RQS of the system, then we can also perform the analysis applying a number of Lyapunov functions in the same way as it was mentioned in Case 1.

3. P is completely undetermined (unknown)

In the case when we have no information and no restrictions on P, we usually look for an optimal (maximal) estimate of P. If we specify a shape class $P(\boldsymbol{\alpha})$ of our estimate, then we can easily obtain for any positive definite matrix $\mathbf{S}$ a partial estimate of P in the form of the sum

$$P_{\mathbf{S}} = \bigcup_{\boldsymbol{\alpha}\in Q(\mathbf{S})} P(\boldsymbol{\alpha}), \tag{3.4}$$

in the same way as in Case 2.

Similarly to Case 2 we can use suitable measures of $P_{\mathbf{S}}$ in order to perform an optimization. By analogy, we introduce certain measures which have a clear meaning and are of practical importance. Namely:

$$\mu_i(P_{\mathbf{S}}) = \sup_{\alpha_i} \left\{\alpha_i \in R^1_+ \,:\, \gamma(\mathbf{S}, (0, \ldots, 0, \alpha_i, 0, \ldots, 0) > 0\right\} \tag{3.5}$$

for $i = 1, 2, \ldots, l$, and

$$\mu_0(P_{\mathbf{S}}) = \mu_1(P_{\mathbf{S}}) \cdot \mu_2(P_{\mathbf{S}}) \cdot \ldots \cdot \mu_l(P_{\mathbf{S}}). \tag{3.6}$$

It is easy to see that measure μ_i determines simply the size of $P_{\mathbf{S}}$ in i-th base direction in R^l_+ while μ_0 is a kind of an average size of $P_{\mathbf{S}}$. We will use the above measures in the next section.

If a suitable measure μ of $P_{\mathbf{S}}$ is chosen, then one can calculate an optimal matrix $\widehat{\mathbf{S}}$ from the optimization problem

$$\sup_{\mathbf{S}} \mu(P_{\mathbf{S}}) \tag{3.7}$$

unless $\mathbf{S}$ is fixed. If such a matrix exists then the system is robustly exponentially stable with respect to the norm $\|\cdot\|_{\widehat{\mathbf{S}}}$ in the optimal set of parameters of the maximal measure μ.

It is quite clear how to generalize the above optimization procedure to the case when various shape classes $P_1(\boldsymbol{\alpha})$, $P_2(\boldsymbol{\alpha})$, ... are used simultaneously to estimate P. The main question arising here is what classes of shapes and how many should be used in order to obtain a relatively simple and accurate estimate of the stability set. In practice we have at least one shape class distinguished by our general expectations concerning a given uncertain system. Moreover, in many real systems the uncertainties $z_1, \ldots, z_k$ are independent and bounded (i.e. $|z_i| \leq \alpha_i$ for certain $\alpha_i \in R_+$, $i = 1, \ldots, l$). In such cases, the rectangular shape class $P(\boldsymbol{\alpha}) = \{(q_1, \ldots, q_l) : |q_i| \leq \alpha_i, i = 1, \ldots, l\}$ is naturally distinguished and very convenient for stability analysis.

On the other hand, if we are interested in RQS only, we can apply various measures of $P_{\mathbf{S}}$ simultaneously. Then, in the result of the optimization with respect to the measures we obtain, in general, various optimal matrices $\widehat{\mathbf{S}}_1$, $\widehat{\mathbf{S}}_2, \ldots, \widehat{\mathbf{S}}_j, \ldots$ and an optimal and open estimate of the RQS set in the form of the sum

$$\widehat{P} = \bigcup_j P_{\widehat{\mathbf{S}}_j}. \tag{3.8}$$

Similarly, an open question is here of what measures should be used and how many in order to obtain a useful estimate of the stability set. Having in mind Theorem 3 we may expect that a finite number of suitable measures will be sufficient for obtaining relatively accurate estimates. We hope that measures (3.5), (3.6) are sufficient for obtaining satisfactory results of the robust stability analysis in many practical cases. The usefulness and simplicity of these measures will be demonstrated in the next section.

4. Robust stability of affine systems

We apply in this section the presented method of stability analysis to the problem of robust stability of an affine nonautonomous system of the following form

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + z_1(t)\mathbf{f}_1(\mathbf{x}) + \ldots + z_k(t)\mathbf{f}_k(\mathbf{x}), \tag{4.1}$$

where $\mathbf{x} \in R^n$, $\mathbf{f}_i : R^n \to R^n$, $\mathbf{f}_i(\mathbf{0}) = \mathbf{0}$, $i = 0, \ldots, k$ and $z_1, \ldots z_k$ are scalar perturbations such that $|z_i(t)| \leq \alpha_i$, $i = 1, \ldots, k$ for certain bounds $\alpha_1, \ldots, \alpha_k \in R_+$ and for every $t \geq t_0$. We assume that all perturbations z_i, $i = 1, \ldots, k$ are independent, i.e. each perturbation z_i adds its own independent contribution $z_i(t)\mathbf{f}_i(\mathbf{x})$ to the system dynamics. No more information about the perturbations is assumed to be known. Therefore, as it has been noted in Sec. 3, there is a naturally distinguished rectangular shape class $P(\boldsymbol{\alpha}) = \{\mathbf{z} \in R^k : |z_1| \leq \alpha_1 \wedge \ldots \wedge |z_k| \leq \alpha_k\}$ which we apply in further considerations for convenience. We also assume that the stationary part of the system (4.1), i.e. the stationary system $\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x})$ is asymptotically stable in R^n.

The above assumptions determine a wide class of uncertain systems so that many nonstationary uncertain systems that are ecountered in practice can be described by equations of the form (4.1). In particular, any system (1.2), with differentiable function $\mathbf{f}$ and with sufficiently small perturbations $\mathbf{z} = (z_1, \ldots, z_k)^T$ acting according to (2.2), can be approximated by the system of the type (4.1) with $\mathbf{f}_0(\mathbf{x}) = \mathbf{f}(\mathbf{x}, \mathbf{p}_0)$, $\mathbf{f}_i(\mathbf{x}) = \partial\mathbf{f}(\mathbf{x}, \mathbf{p}_0)/\partial p_i$, $i = 1, \ldots, k$.

It is easily seen that the problem of stability of (4.1) is equivalent to the problem of SRS of the uncertain system $(\mathbf{f}, P)$, where

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}_0(\mathbf{x}) + p_1\mathbf{f}_1(\mathbf{x}) + \ldots + p_k\mathbf{f}_k(\mathbf{x}) \tag{4.2}$$

and $\mathbf{p} = (p_1, \ldots, p_k) \in P(\boldsymbol{\alpha}) = \{\mathbf{q} \in R^k : \ q_i \leq \alpha_i, \ i = 1, \ldots, k\}$.

In order to apply the results obtained in previous sections we perform the SRNS and WRQS analysis of the system. According to (2.9), the stability index $\widehat{\gamma}(\mathbf{S}, \mathbf{p})$ of the system for any $\mathbf{p} \in P(\boldsymbol{\alpha})$ can be calculated from the following formula

$$\widehat{\gamma}(\mathbf{S}, \mathbf{p}) = -\sup_{\mathbf{x} \neq \mathbf{0}} \Big[g_0(\mathbf{x}) + p_1 g_1(\mathbf{x}) + \ldots + p_k g_k(\mathbf{x})\Big], \tag{4.3}$$

where $g_i(\mathbf{x}) = \mathbf{x}^T \mathbf{S}\, \mathbf{f}_i(\mathbf{x})/\|\mathbf{x}\|^2_{\mathbf{S}}$. To ensure that $\widehat{\gamma}(\mathbf{S}, \mathbf{p}) < \infty$ for $\mathbf{p} \in P(\boldsymbol{\alpha})$ we assume that

$$\mathop{\forall}_{i=1,\ldots,k} \ \mathop{\exists}_{\|\cdot\|_i} \ \mathop{\exists}_{h_i > 0} \ \mathop{\forall}_{\mathbf{x} \in R^n} \ \|\mathbf{f}_i(\mathbf{x})\|_i \leq h_i \|\mathbf{x}\|_i . \tag{4.4}$$

Obviously, if $\forall(\mathbf{x} \in R^n)$, $\|\mathbf{f}_i(\mathbf{x})\|_i \leq h_i\|\mathbf{x}\|_i$ for a certain norm $\|\cdot\|_i$, then the analogous inequality with a suitable finite constant h is valid for any other equivalent norm $\|\cdot\|$ in R^n. Moreover, for any positive definite matrix $\mathbf{S}$

$$\delta_i(\mathbf{S}) = \sup_{\mathbf{x} \neq \mathbf{0}} |g_i(\mathbf{x})| < \infty, \qquad i = 1, \ldots, k. \tag{4.5}$$

In fact, applying Schwartz inequality and the theorem on equivalence of norms in R^n we obtain

$$|g_i(\mathbf{x})| = |\mathbf{x}^T \mathbf{S}\, \mathbf{f}_i(\mathbf{x})|/\|\mathbf{x}\|^2_{\mathbf{S}} \leq \|\mathbf{x}\|_{\mathbf{S}} \|\mathbf{f}(\mathbf{x})\|_{\mathbf{S}}/\|\mathbf{x}\|^2_{\mathbf{S}} \leq h_i(\mathbf{S}) = \text{const} < \infty$$

for every $\mathbf{x} \neq \mathbf{0}$. Hence, also $\delta_0(\mathbf{S}) < \infty$.

It is important for our purposes that the uncertain systems of the considered class has the continuity property, i.e. the index $\widehat{\gamma}(\mathbf{S},\mathbf{p})$ is a continuous function of $\mathbf{p}$.

THEOREM 5. *Any uncertain system* $(\mathbf{f}, P)$ *with* $P = P(\boldsymbol{\alpha})$, $\mathbf{f}$ *given by* (4.2) *and satisfying* (4.4), *has the continuity property, i.e. the index* $\widehat{\gamma}(\mathbf{S},\mathbf{p})$ *defined by* (4.3) *is a continuous function of parameters* $\mathbf{p}$.

Proof. Let $\mathbf{p}$, $\mathbf{p}^*$ be two vectors of parameters belonging to $P(\alpha)$ and suppose, for instance, that $\widehat{\gamma}(\mathbf{S},\mathbf{p}) > \widehat{\gamma}(\mathbf{S},\mathbf{p}^*)$. Then

$$|\widehat{\gamma}(\mathbf{S},\mathbf{p}) - \widehat{\gamma}(\mathbf{S},\mathbf{p}^*)| = \left|\sup_{\mathbf{x}\neq\mathbf{0}} [g_0(\mathbf{x}) + p_1^* g_1(\mathbf{x}) + \ldots + p_k^* g_k(\mathbf{x})] + \widehat{\gamma}(\mathbf{S},\mathbf{p})\right| \leq \ldots$$
$$\leq \left|\sup_{\mathbf{x}\neq\mathbf{0}} [(p_1^* - p_1)g_1(\mathbf{x}) + \ldots + (p_k^* - p_k)g_k(\mathbf{x})]\right| \leq \ldots$$
$$\leq |p_1^* - p_1| \sup_{\mathbf{x}\neq\mathbf{0}} |g_1(\mathbf{x})| + \ldots + |p_k^* - p_k| \sup_{\mathbf{x}\neq\mathbf{0}} |g_k(\mathbf{x})|.$$

Thus, applying (4.5), we conclude that there exist positive constants $\delta_i = \delta_i(\mathbf{S})$, $i = 1, \ldots, k$ such that

$$|\widehat{\gamma}(\mathbf{S},\mathbf{p}) - \widehat{\gamma}(\mathbf{S},\mathbf{p}^*)| \leq \delta_1|p_1 - p_1^*| + \ldots + \delta_k|p_k - p_k^*|$$

for every $\mathbf{p}, \mathbf{p}^* \in P(\alpha)$. This completes the proof.

In the same way one can prove the following

THEOREM 6. *The stability index* $\widehat{\gamma}(\mathbf{S},\boldsymbol{\alpha})$ (*calculated according to* (2.12)) *for system* (4.1) *with* $P = P(\boldsymbol{\alpha})$ *and functions* $\mathbf{f}_i$, $i = 1, \ldots, k$ *satisfying* (4.4), *is a continuous function of* $\alpha_1, \ldots, \alpha_k \in R_+$.

Therefore we can apply to the system under consideration the results obtained in Secs. 2 and 3.

It is clear that in order to achieve the stability of the system (4.1), we have to assume that the index

$$\gamma_0(\mathbf{S}) = \widehat{\gamma}(\mathbf{S},\mathbf{0}) = -\sup_{\mathbf{x}\neq\mathbf{0}} g_0(\mathbf{x}) \tag{4.6}$$

is positive, i.e. the system (4.1) without uncertainties ($\boldsymbol{\alpha} = \mathbf{0}$) should be exponentially stable with respect to a norm $\|\cdot\|_{\mathbf{S}}$. Then, it follows from the continuity of $\widehat{\gamma}(\mathbf{S},\boldsymbol{\alpha})$ that the index $\widehat{\gamma}(\mathbf{S},\boldsymbol{\alpha})$ remains positive locally in a neighbourhood of $\boldsymbol{\alpha} = \mathbf{0}$, i.e. system (4.1) remains stable for sufficiently small perturbation bounds $\alpha_1, \ldots, \alpha_k$.

If the bounds $\alpha_1, \ldots, \alpha_k$ are known and fixed, then we can only check whether $\hat{\gamma}(\mathbf{S}, \boldsymbol{\alpha}) > 0$, i.e. whether the system is strongly robustly exponentially stable in $P(\boldsymbol{\alpha})$ with respect to a given norm $\|\cdot\|_{\mathbf{S}}$. We can also perform a Lyapunov function optimization, for example (3.1).

Since the index $\hat{\gamma}(\mathbf{S}, \boldsymbol{\alpha})$ given by (2.12) can rarely be derived exactly as a function of $\boldsymbol{\alpha}$, we can apply the following obvious estimate

$$(4.7) \qquad \hat{\gamma}(\mathbf{S}, \boldsymbol{\alpha}) \geq \gamma(\mathbf{S}, \boldsymbol{\alpha}) = \gamma_0(\mathbf{S}) - \alpha_1 \delta_1(\mathbf{S}) - \ldots - \alpha_k \delta_k(\mathbf{S}).$$

The above estimate is especially useful in the case when the size or the shape of P for system (4.1) is not known. Since the approximate index $\gamma(\mathbf{S}, \boldsymbol{\alpha})$ is a linear function of $\boldsymbol{\alpha}$, it is relatively easy to find the set $Q(\mathbf{S})$ and an estimate of P for the natural rectangular shape class $P(\boldsymbol{\alpha})$. Indeed, the set $P_{\mathbf{S}}$ defined by (3.4) is, in this case, a simple polyhedron with $k+1$ vertices: $(0, \ldots, 0)$, $(\gamma_0(\mathbf{S})/\delta_1(\mathbf{S}), 0, \ldots, 0), \ldots, (0, \ldots, 0, \gamma_0(\mathbf{S})/\delta_k(\mathbf{S}))$. We can optimise $P_{\mathbf{S}}$ with respect to various measures e.g. those proposed in Sec. 3. According to general definitions (3.5), (3.6), we obtain in our case

$$(4.8) \qquad \mu_0(P_{\mathbf{S}}) = \frac{[\gamma_0(\mathbf{S})]^k}{\delta_1(\mathbf{S}) \cdot \ldots \cdot \delta_k(\mathbf{S})},$$

$$(4.9) \qquad \mu_i(P_{\mathbf{S}}) = \gamma_0(\mathbf{S})/\delta_i(\mathbf{S}), \qquad i = 1, \ldots, k.$$

The presented analysis becomes quite simple in the case of uncertain linear systems of the form

$$(4.10) \qquad \dot{\mathbf{x}} = \mathbf{A}_0 \mathbf{x} + z_1(t) \mathbf{A}_1 \mathbf{x} + \ldots + z_k(t) \mathbf{A}_k \mathbf{x},$$

where $\mathbf{x} \in R^n$, $\mathbf{A}_i$, $i = 0, \ldots, k$ are real constant $n \times n$ matrices, $\mathbf{A}_0$ is stable, $z_1(t), \ldots, z_k(t)$ are scalar perturbations such that $|z_i(t)| \leq \alpha_i$, $i = 1, \ldots, k$ for certain bounds $\alpha_1, \ldots, \alpha_k \in R^1_+$ and for every $t \geq t_0$ (see, for example [8]).

Since the right-hand side of equations (4.10) is linear both with respect to $\mathbf{x}$ and $\mathbf{z} = [z_1, \ldots, z_k]$, such systems are sometimes called bilinear systems. Hence, system (4.10) remains linear with respect to the state $\mathbf{x}$ in spite of the acting perturbations. Therefore we say that we have to deal with the so-called structural perturbations. Each perturbation $z_i(t)$, $i = 1, 2, \ldots, k$, adds at any time $t > t_0$ its own independent contribution $z_i(t)\mathbf{A}_i$ to the system matrix

$$(4.11) \qquad \mathbf{A}(t) = \mathbf{A}_0 + z_1(t)\mathbf{A}_1 + \ldots + z_k(t)\mathbf{A}_k.$$

In particular, a linear uncertain system

$$(4.12) \qquad \dot{\mathbf{x}} = \mathbf{A}(\mathbf{p})\mathbf{x},$$

with the parameters $\mathbf{p} \in R^n$ perturbed according to (2.2) can be approximated, for sufficiently small perturbations $z(t)$, by a linear system of the matrix (4.11), where $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ and $\mathbf{A}_i = (\partial\mathbf{A}/\partial\mathbf{p})(\mathbf{p}_0)$, $i = 1, \ldots, k$. Then $\mathbf{A}_0$ represents the stable, nominal part of the original system (4.10) while the matrices $\mathbf{A}_i$, $i = 1, \ldots, k$ determine the influence of the corresponding perturbations $z_i(t)$ on the system dynamics.

It is easy to see that the functions $g_i(\mathbf{x}) = \mathbf{x}^T\mathbf{S}\,\mathbf{A}_i\mathbf{x}/\|\mathbf{x}\|^2_{\mathbf{S}}$ for system (4.10) satisfy conditions (4.4). Moreover, according to (2.12) the stability index for the system can be calculated from the formula

$$\gamma(\mathbf{S}) = -\sup_{\mathbf{x}\neq\mathbf{0}} \sup_{\mathbf{z}} \left(\frac{\mathbf{x}^T\mathbf{S}\,\mathbf{A}\,\mathbf{x}_0}{\mathbf{x}^T\mathbf{S}\,\mathbf{x}} + \sum_{i=1}^{k} z_i \frac{\mathbf{x}^T\mathbf{S}\,\mathbf{A}_i\mathbf{x}}{\mathbf{x}^T\mathbf{S}\,\mathbf{x}} \right). \tag{4.13}$$

It follows from (4.13) that the supremum over $\mathbf{z} \in P(\boldsymbol{\alpha})$ is achieved for

$$z_i = \widehat{z}_i(\mathbf{x}) = \alpha_i \operatorname{sign}\left(\mathbf{x}^T\mathbf{S}\,\mathbf{A}_i\mathbf{x}\right), \qquad i = 1, \ldots, k. \tag{4.14}$$

According to our remarks given in Sec. 2, the above formulae can be interpreted as a multiloop feedback applied to the system, as is illustrated in Fig. 2.

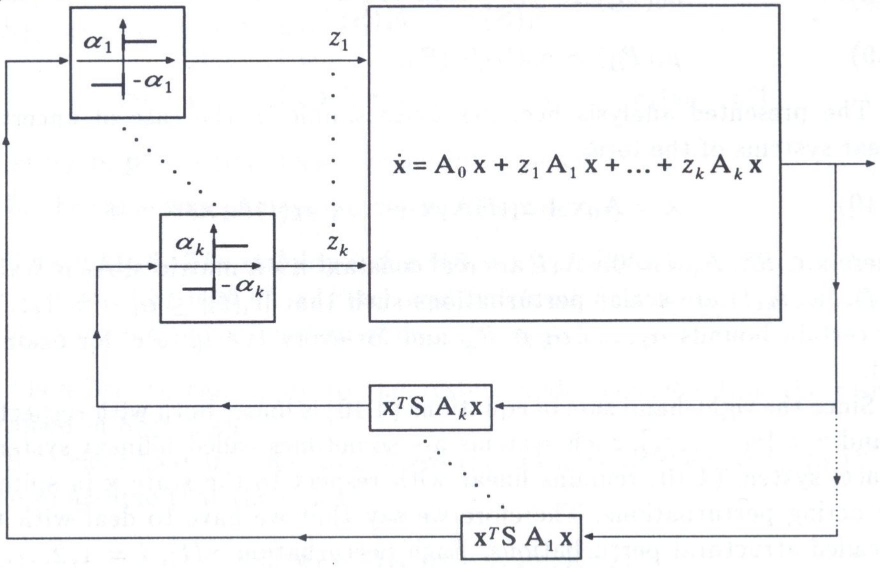

FIG. 2. Block scheme of the equivalent stationary nonlinear closed-loop system.

In this way, the problem of stability of the nonautonomous (uncertain) system (4.10) reduces in this approach to the equivalent problem for a nonlinear but autonomous system (without uncertainties) which can be described by the following equations

$$\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} + \widehat{z}_1(\mathbf{x})\mathbf{A}_1\mathbf{x} + \ldots + \widehat{z}_k(\mathbf{x})\mathbf{A}_k\mathbf{x} \tag{4.15}$$

(see (2.8)). It is seen that (4.15) is a piece-wise linear system, since the most disadvantageous perturbations (4.14) are piece-wise constant functions (more precisely, they are bang-bang functions with quadratic switching surfaces). Note that the equivalent autonomous feedback system (4.15) is essentially nonlinear in spite of linearity of the origin system (4.10).

To find the optimal stability index $\widehat{\gamma}$, supremum (4.13) over $\mathbf{x} \neq \mathbf{0}$ should be calculated for perturbations given by (4.14). Therefore, formula (4.13) can be rewritten as follows

$$\widehat{\gamma}(\mathbf{S}) = -\sup_{\mathbf{x}\neq\mathbf{0}} \left(\frac{\mathbf{x}^T\mathbf{S}\,\mathbf{A}_0\mathbf{x}}{\mathbf{x}^T\mathbf{S}\,\mathbf{x}} + \sum_{i=1}^{k} \alpha_i \frac{|\mathbf{x}^T\mathbf{S}\,\mathbf{A}_i\mathbf{x}|}{\mathbf{x}^T\mathbf{S}\,\mathbf{x}} \right). \tag{4.16}$$

The above supremum still is not easy for calculations in the multidimensional case. However, it is possible to obtain analytical results by using estimate (4.7). In the case of linear uncertain system (4.10), we have the following simple formula

$$\gamma_0(\mathbf{S}) = -\sup_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{S}\,\mathbf{A}_0\mathbf{x}}{\mathbf{x}^T\mathbf{S}\,\mathbf{x}} \tag{4.17}$$

representing exponential rate of convergence of the system without perturbations (see [3, 11]) and the formulae

$$\delta_i(\mathbf{S}) = \sup_{\mathbf{x}\neq\mathbf{0}} \left| \frac{\mathbf{x}^T\mathbf{S}\,\mathbf{A}_i\mathbf{x}}{\mathbf{x}^T\mathbf{S}\,\mathbf{x}} \right|, \qquad i = 1, \ldots, k. \tag{4.18}$$

It is obvious that $\delta_i(\mathbf{S}) > 0$ for $i = 1, \ldots, k$.

It is shown in the Appendix how the quantities (4.17), (4.18) can be calculated. For a given matrix $\mathbf{S}$ they usually cannot be expressed explicitly as functions of $\mathbf{S}$ although they can be calculated numerically by standard methods. However, thanks to our estimate $\gamma(\mathbf{S}, \boldsymbol{\alpha}) \leq \widehat{\gamma}(\mathbf{S}, \boldsymbol{\alpha})$, the approximate index (4.7) is a simple linear function of the parameters $\alpha_1, \ldots, \alpha_k$. This makes it possible to obtain certain approximate but analytical results which are of practical importance.

Performing the optimization with respect to the introduced quality indices μ_i, $i = 0, 1, \ldots, k$, one obtains optimal matrices $\widehat{\mathbf{S}}_0, \ldots, \widehat{\mathbf{S}}_k$, respectively, and the corresponding optimal Lyapunov functions. On this basis one can find an optimal estimate (3.8) of the stability set

$$\bigcup_{i=0}^{k} \left\{ (\alpha_1, \ldots, \alpha_k) \in R_+^k : \ \gamma_0(\widehat{\mathbf{S}}_i) - \alpha_1\delta_1(\widehat{\mathbf{S}}_i) - \ldots - \alpha_k\delta_k(\widehat{\mathbf{S}}_i) \geq 0 \right\} \tag{4.19}$$

confined to $k+1$ hyperplanes in R^k.

5. Robust stability of an uncertain oscillator

As an example, we consider in this section an oscillator described by the second order equation

$$\ddot{x}_1 + 2p_1\dot{x}_1 + p_2 x_1 = 0, \tag{5.1}$$

where $x_1 \in R$, and p_1, p_2 are uncertain parameters of damping and rigidity, respectively. We assume the following decompositions

$$\mathbf{p} = (p_1, p_2) = (1 + z_1(t),\ 1 + z_2(t)) = \mathbf{p}_0 + \mathbf{z}, \tag{5.2}$$

where $\mathbf{p}_0 = (1,1)$ represents the nominal values of the parameters and $\mathbf{z} = (z_1, z_2)$ are their perturbations. We also assume that the perturbations are independent and bounded, i.e.

$$\underset{\alpha_i \geq 0}{\exists}\ \underset{t \geq t_0}{\forall}\ \|z_i(t)\| \leq \alpha_i, \qquad i = 1,\ 2.$$

Thus the vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in R^2_+$ determines the perturbation bounds.

It is easy to write equation (5.1) in the matrix form (4.10) with $k = 2$, $\mathbf{x} = [x_1, \dot{x}_1]^T = [x_1, x_2]^T$ and

$$\mathbf{A}_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \qquad \mathbf{A}_1 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

Thus the robust stability of the oscillator (5.1) can be studied by using the method described in the previous section. Since the particular cases: ($\alpha_1 = 0$, $\alpha_2 > 0$), ($\alpha_1 \geq 0, \alpha_2 = 0$) of such an oscillator have been considered in details in many papers (for example in [3, 9–10]), we concentrate here on the case of two nonvanishing independent perturbations when $\alpha_1 > 0$, $\alpha_2 > 0$ (see. e.g. [14]).

According to (6.5), (6.8), (6.9), the exponential rate of convergence for our oscillator is equal to

$$\gamma(\mathbf{S}, \boldsymbol{\alpha}) = \min[\gamma_+, \gamma_-],$$

where

$$\mathbf{S} = \begin{bmatrix} a & b \\ b & 1 \end{bmatrix}, \qquad a - b^2 > 0$$

and

$$\gamma_{\pm}(\mathbf{S}) = 1 \pm \alpha_1 - \sqrt{[b - (1 \pm \alpha_1)]^2 + \frac{(|a - 2b^2 + 2b(1 \pm \alpha_1) - 1| + \alpha_2)^2}{4(a - b^2)}}. \tag{5.3}$$

Looking at (5.3) it is seen that it is worth, for simplicity, to restrict the class of Lyapunov functions by the relation (see [3])

$$a = 2b^2 - 2b + 1, \tag{5.4}$$

where $b \in (1,2)$. Under such an assumption all quantities dependent on the matrix $\mathbf{S}$ are in fact functions of a scalar variable $b \in (1,2)$. For this reason the Lyapunov function optimization can be easily performed.

According to general descriptions given in Sec. 4 it can be calculated that

$$\mu_1(\mathbf{S}) = \alpha_{1\,\max}(b) = (b-1)(2-b)/b,$$
$$\mu_2(\mathbf{S}) = \alpha_{2\,\max}(b) = 2(b-1)\sqrt{b(2-b)}$$

under assumption (5.4). The optimization (maximization) performed independently with respect to $\mu_0(\mathbf{S}) = \alpha_{1\,\max}(b)\cdot\alpha_{2\,\max}(b)$ and $\mu_i(\mathbf{S}) = \alpha_{i\,\max}(b)$, $i = 1, 2$, over $b \in (1,2)$, gives the following parameters of optimal Lyapunov functions

$$\begin{aligned} &\widehat{b}_0 &&\cong 1.5486, \qquad &&\widehat{a}_0 &&\cong 2.6991,\\ &\widehat{b}_1 = \sqrt{2} &&\cong 1.4142, &&\widehat{a}_1 = 5 - 2\sqrt{2} &&\cong 2.1715,\\ &\widehat{b}_2 = 1 + \sqrt{2}/2 &&\cong 1.7071, &&\widehat{a}_2 = 2 + \sqrt{2} &&\cong 3.4142, \end{aligned}$$

and optimal values of the measures:

$$\mu_0(\widehat{\mathbf{S}}_0) \cong 0.1467, \qquad \mu_1(\widehat{\mathbf{S}}_1) = 3 - 2\sqrt{2} \cong 0.1715, \qquad \mu_2(\widehat{\mathbf{S}}_2) = 1,$$

respectively.

Applying the above results one can easily deduce from (5.3), (5.4) that an estimate of the stability set of the oscillator in the space of parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in R^2_+$ can be described by the following inequalities (compare with the result obtained in [14])

$$4\widehat{b}_i(2 - \widehat{b}_i - 2\alpha_1)(\widehat{b}_i^2 - 1)^2 > (2\widehat{b}_i\alpha_1 + \alpha_2)^2, \qquad i = 0,\ 1,\ 2. \tag{5.5}$$

It is easy to see that the obtained estimate in R^2_+ is bounded by three parabolas. Applying approximate index (4.7) for the analysis and formulae (4.17), (4.18) and performing an optimization e.g. with respect measures (4.8), (4.9), one can obtain more conservative stability region of the form (4.19) bounded by straight lines. On the other hand, one can improve the result (5.5) by performing the optimization in a two-parametric class of Lyapunov functions (i.e. without restriction (5.4)).

Appendix

We will prove in this Appendix some useful formulae that enable us to express stability indices of uncertain linear systems in terms of eigenvalues of certain symmetric matrices.

Let $\mathbf{A}$, $\mathbf{C}$, $\mathbf{S}$, $\mathbf{D}$ be constant, real matrices $n \times n$, such that $\mathbf{C}$, $\mathbf{S}$ are symmetric ones, $\mathbf{S}$ is positive definite and $\mathbf{D} = \sqrt{\mathbf{S}}$. Let us also denote by $\lambda_{\min}(\mathbf{C})$, $\lambda_{\max}(\mathbf{C})$ the minimal and the maximal eigenvalue of the matrix $\mathbf{C}$, respectively. Then one can prove the following

PROPOSITION 1 (see [3, 5, 9, 11])

$$-\sup_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T\mathbf{S}\,\mathbf{A}\,\mathbf{x}}{\mathbf{x}^T\mathbf{S}\,\mathbf{x}} = \lambda_{\min}(\mathbf{C}),$$

where

$$\mathbf{C} = -\frac{1}{2}(\mathbf{D}\,\mathbf{A}\,\mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{A}^T\mathbf{D}). \tag{6.1}$$

Proof

$$-\sup_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T\mathbf{S}\,\mathbf{A}\,\mathbf{x}}{\mathbf{x}^T\mathbf{S}\,\mathbf{x}} = -\sup_{\mathbf{x}^T\mathbf{S}\,\mathbf{x}=1} \frac{\mathbf{x}^T\mathbf{D}^T(\mathbf{D}\,\mathbf{A}\,\mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{A}^T\mathbf{D})\mathbf{D}\,\mathbf{x}}{2\mathbf{x}^T\mathbf{D}^T\mathbf{D}\,\mathbf{x}}$$

$$= -\sup_{\mathbf{y}^T\mathbf{y}=1} \mathbf{y}^T(-\mathbf{C})\mathbf{y} = -\lambda_{\max}(-\mathbf{C}) = \lambda_{\min}(\mathbf{C}).$$

Similarly one can prove

PROPOSITION 2

$$\sup_{\mathbf{x}\neq 0} \frac{|\mathbf{x}^T\mathbf{S}\,\mathbf{A}\,\mathbf{x}|}{\mathbf{x}^T\mathbf{S}\,\mathbf{x}} = \max\Big[|\lambda_{\min}(\mathbf{C})|, |\lambda_{\max}(\mathbf{C})|\Big],$$

where $\mathbf{C}$ is given by (6.1).

Proof

$$\sup_{\mathbf{x}\neq 0} \frac{|\mathbf{x}^T\mathbf{S}\,\mathbf{A}\,\mathbf{x}|}{\mathbf{x}^T\mathbf{S}\,\mathbf{x}} = \sup_{\mathbf{x}^T\mathbf{S}\,\mathbf{x}=1} \left[\frac{|\mathbf{x}^T\mathbf{D}^T(\mathbf{D}\,\mathbf{A}\,\mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{A}^T\mathbf{D})\mathbf{D}\,\mathbf{x}|}{\mathbf{x}^T\mathbf{D}^T\mathbf{D}\,\mathbf{x}}\right]$$

$$= \sup_{\mathbf{y}^T\mathbf{y}=1} |\mathbf{y}^T(-\mathbf{C})\mathbf{y}| = \max\left[\left|\inf_{\mathbf{y}^T\mathbf{y}=1} \mathbf{y}^T(-\mathbf{C})\mathbf{y}\right|, \left|\sup_{\mathbf{y}^T\mathbf{y}=1} \mathbf{y}^T(-\mathbf{C})\mathbf{y}\right|\right]$$

$$= \max\left[|\lambda_{\min}(-\mathbf{C})|, |\lambda_{\max}(-\mathbf{C})|\right] = \max\left[|\lambda_{\min}(\mathbf{C})|, |\lambda_{\max}(\mathbf{C})|\right].$$

Now we prove that in the general multidimensional case, the exponential rate of convergence (4.16) of a given uncertain linear system can be expressed in terms of eigenvalues of certain matrices (see [5]). First, let us introduce, for a binary sequence $r_1, \ldots, r_k$, $r_i \in \{-1, 1\}$, $i = 1, \ldots, k$, the matrix $\mathbf{B}_j = \mathbf{A}_0 + r_1\alpha_1\mathbf{A}_1 + \ldots + r_k\alpha_k\mathbf{A}_k$ and the corresponding characteristic index

$$d_j(\mathbf{S}) = -\sup_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{S}\,\mathbf{B}_j\mathbf{x}}{\mathbf{x}^T\mathbf{S}\,\mathbf{x}},$$

where

$$j = \frac{1}{2} - \left[(r_1 + 2r_2 + \ldots + 2^{k-1}r_k) + (2^k - 1)\right].$$

It is clear that $j = 0, 1, \ldots, 2^k$ for the corresponding series and, according to Proposition 1,

$$d_j(\mathbf{S}) = \lambda_{\min}\left(-\frac{1}{2}\left[\mathbf{D}\,\mathbf{B}_j\mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{B}_j\mathbf{D}\right]\right).$$

Moreover, as one can easily prove, the following proposition is true:

PROPOSITION 3

$$\gamma(\mathbf{S}, \alpha_1, \ldots, \alpha_k) = \min_{j=0,1,\ldots,2^k} [d_j(\mathbf{S})],$$

where the index $\gamma(\mathbf{S}, \alpha_1, \ldots, \alpha_k)$ is given by formula (4.16).

The above formula tells us that the minimal possible speed of exponential convergence of trajectories of the system is achieved for certain constant perturbations $z_i = r_i\alpha_i$, $i = 1, \ldots, k$, where $r_1, \ldots, r_k$ is a binary sequence (see e.g. [5])

In particular, we have

$$\mathbf{B}_0 = \mathbf{A}_0 - \alpha_1\mathbf{A}_1, \qquad \mathbf{B}_1 = \mathbf{A}_0 + \alpha_1\mathbf{A}_1, \qquad \gamma(\mathbf{S}, \alpha_1) = \min[d_0(\mathbf{S}),\ d_1(\mathbf{S})]$$

for $k = 1$.

Similarly, for $k = 2$, there are four matrices:

$$\mathbf{B}_0 = \mathbf{A}_0 - \alpha_1\mathbf{A}_1 - \alpha_2\mathbf{A}_2, \qquad \mathbf{B}_1 = \mathbf{A}_0 + \alpha_1\mathbf{A}_1 - \alpha_2\mathbf{A}_2,$$
$$\mathbf{B}_2 = \mathbf{A}_0 - \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2, \qquad \mathbf{B}_3 = \mathbf{A}_0 + \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2$$

and the stability index

$$\gamma(\mathbf{S}, \alpha_1, \alpha_2) = \min[d_0(\mathbf{S}), d_1(\mathbf{S}), d_2(\mathbf{S}), d_3(\mathbf{S})].$$

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