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A STUDY OF THE NUMERICAL CONVERGENCE OF RAYLEIGH-RITZ METHOD FOR THE FREE VIBRATIONS OF CANTILEVER BEAM OF VARIABLE CROSS-SECTION WITH TIP MASS

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A numerical study on the convergence properties of the Rayleigh–Ritz method is presented, for the dynamic analysis of beams with continuously varying cross-section. The beam is assumed to be slender, the Euler–Bernoulli hypotheses are accepted, and some particular cases are considered, for which a closed-form solution is available in terms of Bessel functions. The comparisons between exact and approximate results can give some hint about the usefulness of the approximate method in more complex situations, for which the exact solution is not attainable.

NOTATION

A cross-sectional area of the beam,
$\mathbf{A}$ matrix, Eq. (2.20),
a_i coefficients of trial functions,
b width of the beam,
B_h defined in Eq. $(3.2)_1$,
C_h defined in Eq. $(3.2)_1$,
C_i constants,
C_R defined in Eq. (2.16),
d nondimensional parameter, Eq. (2.8),
d^* eccentricity of tip mass,
E modulus of elasticity,
$G(\xi)$ function defined in Eq. (3.4),
h depth of the beam,
$H(\xi)$ function defined in Eq. (3.4),
I second moment of cross-sectional area,
I_n, J_n Bessel function of first kinds,
J_M second moment of tip mass,
k nondimensional parameter, Eq. (2.8),
k_{ij} elements of stiffness matrix,
k_R stiffness of the rotational spring,
L length of the beam,
M tip mass,

m_{ij} elements of mass matrix,
m_T total mass of the beam,
n coefficient of tapered cross-section,
N number of trial functions,
p frequency parameter,
q_a frequency parameter,
q_i constants, Eq. (3.1),
$r(x)$ weight function,
w transverse displacement,
x, y Cartesian coordinates,
Z defined in Eq. (2.9).

Greek letters

α taper ratio $= h_2/h_1$,
β nondimensional length,
ϕ trial function defined in Eq. (3.1),
ξ nondimensional length $= x/L$,
Π functional of the problem,
μ nondimensional parameter, Eq. (2.8),
ϱ mass density,
η nondimensional length,
ω natural frequency of the beam,
ζ taper ratio $= b_2/b_1$.

1. Introduction

One of the most frequently used structural models is the cantilever beam with non-uniform cross-section. In fact, a great number of structural bearing systems can be approximately reduced to this simplified scheme, so that it is quite important to deduce exact and approximate methods of analysis which can correctly reproduce the dynamic behaviour of this beam.

Exact analyses can be performed, mainly for single-span, nonuniform beams in the presence of various boundary conditions [1, 2] by expressing the solution in terms of Bessel functions, or by using the Frobenius method, or else by means of the transfer matrix approach, as recently developed by Tan *et al.* [24]. On the other hand, the variational approach has given excellent results for stepped beams, beams with intermediate constraints and more complex systems [3, 4, 18]. For example, Laura *et al.* [14] employed an optimized version of the Rayleigh quotient [10], in order to deduce close upper bounds to the fundamental free vibration frequency of a cantilever beam with eccentric tip mass. More recently, an accurate Rayleigh - Ritz solution has been obtained by Grossi *et al.*, even for higher frequencies, by using the Gram-Schmidt procedure [13, 23] in order to deduce a set of orthogonal polynomials which can be conveniently used as trial functions. In this way all the differentiations and integrations are greatly simplified with respect to other, similar approaches [3, 8].

In this paper the dynamic behaviour of beams with continuously varying cross-section is examined, following two complementary approaches. In the first case an exact method is employed, solving the differential equation of motion in terms of Bessel functions, whereas in the second case a Rayleigh-Ritz procedure is adopted, using as trial functions the orthogonal polynomials suggested in [13].

As an example, a cantilever beam with varying rectangular cross-section has been used, as sketched in Fig. 1. The same structure has already been studied in [14] by means of the optimized Rayleigh method, so that useful numerical comparisons can be shown, in terms of nondimensional parameters.

2. Exact approach

Let us consider the variable cross-section beam shown in Fig. 1 obtained by assembling, in a continuous manner, a constant cross-section beam with a linearly varying cross-section beam. If the properties of the beam are

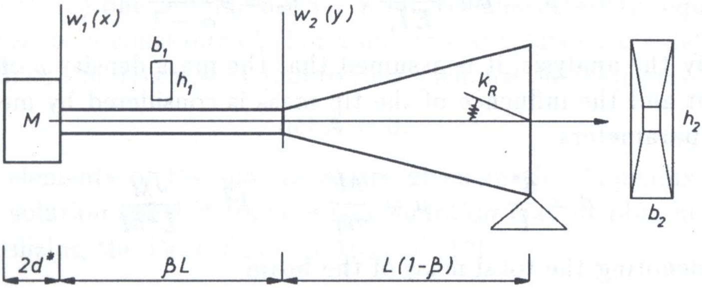

FIG. 1. Sketch of the beam under study.

compatible with Euler-Bernoulli beam theory, and the beam is undergoing a small amplitude free bending vibration, the differential equations may be written in the form

$$\text{(2.1)} \qquad \frac{d^4 w_1(x)}{dx^4} + \frac{\varrho A_1}{EI_1}\omega^2 w_1(x) = 0, \quad 0 \le x \le \beta L,$$

$$\text{(2.2)} \qquad \frac{d^2}{dy^2}\left[EI_y(y)\frac{d^2 w_2(y,t)}{dy^2}\right] + \varrho A_y(y)\frac{d^2 w_2(y,t)}{dy^2} = 0, \quad 0 \le y \le (1-\beta)L,$$

where $w_1(x)$, $w_2(y)$ are the transverse displacements of the beam axis. After introducing the dimensionless variables

$$\xi = \frac{x}{L}, \qquad \eta = 1 + \frac{\alpha - 1}{L(1-\beta)} y, \tag{2.3}$$

the cross-sectional area and the moment of inertia for $0 \le y \le L(1-\beta)$ are given by

$$A_y(y) = A_1 \eta^n, \qquad I_y(y) = I_1 \eta^{n+2}, \tag{2.4}$$

where the coefficient n describes the taper of the cross-section. For $n = 1 \Rightarrow$ $(\zeta = b_2/b_1 = 1,\ \alpha = h_2/h_1)$ whereas, for $n = 2 \Rightarrow (\alpha = b_2/b_1 = h_2/h_1)$.

Using the dimensionless variables (2.3), and substituting Eqs. (2.4) into Eq. (2.2), the governing differential equations become

$$\frac{d^4 w_1}{d\xi^4} - p^4 w_1 = 0; \tag{2.5}$$

$$\begin{aligned} &\eta^2 \frac{d^4 w_2}{d\eta^4} + 8\eta \frac{d^3 w_2}{d\eta^3} + 12 \frac{d^2 w_2}{d\eta^2} - q_a^4 w_2 = 0, \quad 1 \le \eta \le a, \quad \text{for } n = 2, \\ &\eta^2 \frac{d^4 w_2}{d\eta^4} + 6\eta \frac{d^3 w_2}{d\eta^3} + 6 \frac{d^2 w_2}{d\eta^2} - q_a^4 w_2 = 0, \quad 1 \le \eta \le a, \quad \text{for } n = 1, \end{aligned} \tag{2.6}$$

where

$$p^4 = \varrho \omega^2 \frac{A_1 L^4}{E I_1}, \qquad q_a = p \frac{1-\beta}{\alpha - 1}. \tag{2.7}$$

To simplify the analysis, it is assumed that the mass density ϱ of the beam is constant and the influence of the tip mass is considered by means of the following parameters

$$d = \frac{d^*}{L}, \qquad \mu = \frac{M}{m_T}, \qquad k^2 = \frac{J_M}{L^2 M}, \tag{2.8}$$

with m_T denoting the total mass of the beam

$$m_T = \varrho A_1 L \left[\beta + (1-\beta) \frac{(n-1)\alpha^2 + \alpha + 1}{n+1} \right] = \varrho A_1 L Z. \tag{2.9}$$

The boundary and continuity conditions, in term of dimensionless variables, are at $x = 0$, $(\xi = 0)$:

$$\frac{d^2 w_1}{d\xi^2} + \left[k^2 + d^2 \right] \mu Z p^4 \frac{d w_1}{d\xi} - \mu Z d p^4 w_1 = 0, \tag{2.10}$$

$$\frac{d^3 w_1}{d\xi^3} + \mu Z d p^4 \frac{d w_1}{d\xi} - \mu Z p^4 w_1 = 0 \tag{2.11}$$

at point $x = \beta L$, $(\xi = \beta)$ and $y = 0$, $(\eta = 1)$:

$$\frac{d^3 w_1}{d\xi^3} - (n+2)\left[\frac{\alpha-1}{1-\beta}\right]^3 \frac{d^2 w_2}{d\eta^2} - \left[\frac{\alpha-1}{1-\beta}\right]^3 \frac{d^3 w_2}{d\eta^3} = 0, \tag{2.12}$$

$$\frac{d^2 w_1}{d\xi^2} - \left[\frac{\alpha-1}{1-\beta}\right]^2 \frac{d^2 w_2}{d\eta^2} = 0, \tag{2.13}$$

$$w_1 = w_2\,, \qquad \frac{dw_1}{d\xi} - \frac{\alpha-1}{1-\beta}\frac{dw_2}{d\eta} = 0 \tag{2.14}$$

at point $y = L(1-\beta)$, $\eta = \alpha$:

$$\frac{dw_2}{d\eta} + \frac{\alpha-1}{1-\beta} C_R \frac{d^2 w_2}{d\eta^2} = 0, \qquad w_2 = 0, \tag{2.15}$$

where $C_R = EI_2/k_R L$.

The general solution to the differential equations (2.5) and (2.6) can be written, respectively, as

$$w_1(\xi) = C_1 \cosh(p\xi) + C_2 \sinh(p\xi) + C_3 \cos(p\xi) + C_4 \sin(p\xi), \tag{2.16}$$

$$w_2(\xi) = \eta^{-n0.5}\left\{C_5 J_n(a) + C_6 Y_n(a) + C_7 I_n(a) + C_8 K_n(a)\right\}, \tag{2.17}$$

where $a = 2q_a\eta^{0.5}$ and J, Y, I, K are Bessel functions of, respectively, the first, second, modified first and modified second kinds.

The C_i are arbitrary constants to be evaluated from the eight conditions in Eqs. (2.10)–(2.15). Upon substituting Eqs. (2.16) and (2.17) into Eqs. (2.10)–(2.15) one obtains an 8×8 matrix characteristic equation for the eight unknown constants C_i. For a non-trivial solution, the determinant of the coefficient matrix is set equal to zero, yielding the frequency equation:

$$\det \mathbf{A} = 0, \tag{2.18}$$

where the elements of the matrix $\mathbf{A}$ are given in the Appendix I. These non-trivial solutions of the characteristic equation may be obtained numerically by utilizing the False Position Method [12].

3. Rayleigh-Ritz method

The so-called Rayleigh-Ritz method is a general procedure to obtain approximate solutions of problems expressed in variational form. The deflection shapes are approximated by a linear combination of suitably chosen functions

$$w(x) \cong \sum_{i=1}^{N} q_i \phi_i(x), \tag{3.1}$$

where the coefficients q_i are constants to be determined, and the functions $\phi_i(x)$ are usually chosen to satisfy the boundary conditions (2.10), (2.11), (2.15) for any choice of the q_i. It is essential to satisfy the geometrical boundary conditions, [9, 13, 14].

The following set of polynomials $\{\phi_1, \phi_2, \ldots \phi_N\}$ is orthogonal on $[0,1]$ with respect to the weight function $r(x)$, and they are constructed by employing the Gram-Schmidt procedure [13, 23]:

$$\phi_1(x) = \sum_i^4 a_i x^i, \qquad \text{with} \quad a_0 = 1,$$
(3.2)
$$\phi_h(x) = (x - B_h)\phi_{h-1}(x) - C_h\phi_{h-2}(x), \qquad \text{for} \quad h \geq 2,$$

where

$$B_h = \frac{\int_0^L x r(x)[\phi_{h-1}(x)]^2 dx}{\int_0^L r(x)[\phi_{h-1}(x)]^2 dx}, \quad C_h = \frac{\int_0^L x r(x)\phi_{h-1}(x)\phi_{h-2}(x) dx}{\int_0^L r(x)[\phi_{h-2}(x)]^2 dx}$$

and $r(x) = A_x$ is a weight function.

With the approximation (3.1), the functional takes the form [14]

$$\text{(3.3)} \quad \Pi = \frac{1}{2}\left\{\int_0^{\beta L} EI_1 \left[\left(\sum_{i=1}^N q_i\phi_i\right)''\right]^2 + \int_{\beta L}^L EI_x \left[\left(\sum_{i=1}^N q_i\phi_i\right)''\right]^2 dx\right\}$$
$$+\frac{k_R}{2}\left[\left(\sum_{i=1}^N q_i\phi_i(L)\right)'\right]^2 + \omega^2 M d^* \left(\sum_{i=1}^N q_i\phi_i(0)\right)'\left(\sum_{i=1}^N q_i\phi_i(0)\right)$$
$$-\frac{\omega^2}{2}\left\{\int_0^{\beta L} \varrho A_1 \left(\sum_{i=1}^N q_i\phi_i\right)^2 dx + \int_{\beta L}^L \varrho A_x \left(\sum_{i=1}^N q_i\phi_i\right)^2 dx\right\}$$
$$-\omega^2\frac{Md^{*2} + J_M}{2}\left(\sum_{i=1}^N q_i\phi_i(0)\right)'^2 - \omega^2\frac{M}{2}\left(\sum_{i=1}^N q_i\phi_i(0)\right)^2,$$

where $(') = d/dx$, and

$$A_x = A_1\left\{\left[\frac{\zeta\beta - 1}{\beta - 1} + \frac{\zeta - 1}{1 - \beta}\xi\right]\left[\frac{\alpha\beta - 1}{\beta - 1} + \frac{\alpha - 1}{1 - \beta}\xi\right]\right\} = A_1 G(\xi),$$
(3.4)
$$I_x = I_1\left\{\left[\frac{\zeta\beta - 1}{\beta - 1} + \frac{\zeta - 1}{1 - \beta}\xi\right]\left[\frac{\alpha\beta - 1}{\beta - 1} + \frac{\alpha - 1}{1 - \beta}\xi\right]^3\right\} = I_1 H(\xi).$$

In terms of the nondimensional parameters (2.8), (2.9) and after some algebra, Eq. (3.3) can be written

$$(3.5)\qquad \Pi = \frac{EI_1}{2L^3}\left\{\int_0^\beta \left(\sum_{i=1}^N q_i\phi_i\right)^{II^2} d\xi + \int_\beta^1 H(\xi)\left(\sum_{i=1}^N q_i\phi_i\right)^{II^2} d\xi\right.$$

$$+\frac{H(1)}{C_R}\left(\sum_{i=1}^N q_i\phi_i(1)\right)^{I^2} - p^4\left[\int_0^\beta \left(\sum_{i=1}^N q_i\phi_i\right)^2 d\xi + \int_\beta^1 G(\xi)\left(\sum_{i=1}^N q_i\phi_i\right)^2\right.$$

$$+\mu Z\left[d^2+k^2\right]\left(\sum_{i=1}^N q_i\phi_i(0)\right)^{I^2} + \mu Z\left(\sum_{i=1}^N q_i\phi_i(0)\right)^2$$

$$\left.\left.-\mu Zd\left(\sum_{i=1}^N q_i\phi_i(0)\right)^I\left(\sum_{i=1}^N q_i\phi_i(0)\right)\right]\right\}$$

with $(I) = d/d\xi$.

The minimum of this functional can be found from the condition

$$(3.6)\qquad \frac{\partial \Pi}{\partial q_i} = 0.$$

Equations (3.6) yield a set of N homogeneous linear equations for the unknown constants q_j, which can be written as:

$$(3.7)\qquad (\mathbf{K} - p^4\mathbf{m})\mathbf{q} = \mathbf{0},$$

where

$$k_{ij} = \int_0^\beta \phi_i''\phi_j''\,d\xi + \int_\beta^1 H(\xi)\phi_i''\phi_j''\,d\xi + \frac{H(1)}{C_R}\phi_i'(1)\phi_j'(1),$$

$$m_{ij} = \int_0^\beta \phi_i\phi_j\,d\xi + \int_\beta^1 G(\xi)\phi_i\phi_j\,d\xi + \mu Z(d^2+k^2)\phi_i'(0)\phi_j'(0)$$

$$+\mu Z\phi_i(0)\phi_j(0) - \mu Zd\left[\phi_i(0)\phi_j'(0) + \phi_j(0)\phi_i'(0)\right].$$

We note that the coefficients are symmetric; $k_{ij} = k_{ji}$, $m_{ij} = m_{ji}$. Moreover, the stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{m}$ are positive definite. Hence, all the eigenvalues p_i are real and positive. Solving Eqs. (3.7), N approximate p_i values can be determined. It is obvious that the accuracy of the Rayleigh - Ritz results depends on the number N of the assumed mode shape functions ϕ_i.

4. Results and discussion

As a first example, the first five nondimensional free vibration frequencies of the beam in Fig. 1 have been calculated, both with the exact approach and the Rayleigh - Ritz method. The results are given in Tables 1 – 3 for an increasing number of coordinate functions, and they confirm that the approximate method gives upper bounds to the true frequencies, and the discrepancies tend to zero for increasing N.

Table 1. Nondimensional coefficients p_i, $i = 1, \dots 5$ for $C_R = 0$, $\alpha = 1.1$, $d = 0$, $\beta = 0.25$, $\mu = 1$, and various k; (E) Exact values, (R-R) Rayleigh-Ritz method, [14] Rayleigh-Schmidt method.

k	p_1	p_2	p_3	p_4	p_5	N	
0	1.310636	4.143697	7.282355	10.458012	13.643175	–	E
	1.310637	4.144217	7.292440	11.096015	17.548356	5	R-R
	1.310636	4.143711	7.286029	10.596085	14.414860	6	
	1.310636	4.143703	7.282489	10.476452	14.024560	7	
	1.260952	–	–	–	–		[14]
0.3	1.255657	2.561903	5.077380	8.139947	11.296547	–	E
	1.255659	2.561926	5.077764	8.244081	11.672985	5	R-R
	1.255658	2.561923	5.077464	8.143291	11.648398	6	
	1.255657	2.561912	5.077910	8.141582	11.314325	7	
	1.212436	–	–	–	–		[14]
0.6	1.125851	2.050211	5.016769	8.125441	11.291127	–	E
	1.125854	2.050223	5.017178	8.228496	11.667264	5	R-R
	1.125853	2.050223	5.016837	8.128815	11.641913	6	
	1.125851	2.050216	5.016779	8.127034	11.308920	7	
	1.100000	–	–	–	–		[14]
0.9	0.993803	1.901521	5.005729	8.122779	11.290127	–	E
	0.993807	1.901530	5.006142	8.225635	11.666209	5	R-R
	0.993806	1.901530	5.005794	8.126158	11.640717	6	
	0.993804	1.901525	5.005739	8.124364	11.307923	7	
	0.979796	–	–	–	–		[14]

For $N = 5$, the first three nondimensional frequencies show a percentage error within 0.6%, and, moreover, this error decreases for increasing k.

For $N = 7$, the approximate upper bounds are always very close to the true nondimensional frequencies, and at the same time the computational cost is acceptable, so that the choice of $N = 7$ seems to be a good compromise between accuracy and feasibility.

Table 2. As Table 1 for $\beta = 0.5$ and various k.

k	p_1	p_2	p_3	p_4	p_5	N R-R
0	1.372476	–	–	–	–	1
	1.302049	4.705504	–	–	–	2
	1.300096	4.154607	8.808805	–	–	3
	1.299995	4.138507	7.373859	13.571768	–	4
	1.299930	4.133170	7.300342	10.762210	19.093173	5
	1.299928	4.132752	7.262593	10.569321	14.396046	6
	1.299917	4.132620	7.258857	10.425858	13.99948	7
	1.299912	4.132569	7.257972	10.403565	13.568826	Exact
0.3	1.84744	–	–	–	–	1 R-R
	1.247127	2.570042	–	–	–	2
	1.245418	2.569443	5.102448	–	–	3
	1.245361	2.567491	5.09940	8.247609	–	4
	1.245297	2.567458	5.076202	8.237415	11.655622	5
	1.245296	2.567408	5.076093	8.118060	11.632402	6
	1.245285	2.567405	5.075519	8.117109	11.261233	7
	1.245279	2.567397	5.075449	8.113387	11.237672	Exact
0.6	1.125697	–	–	–	–	1 R-R
	1.118304	2.056142	–	–	–	2
	1.117102	2.055782	5.040611	–	–	3
	1.117092	2.054599	5.036272	8.232502	–	4
	1.117035	2.054599	5.014063	8.221342	11.649681	5
	1.117035	2.054568	5.013914	8.103149	11.625680	6
	1.117025	2.054568	5.013372	8.102130	11.255613	7
	1.1170191	2.054565	5.013301	8.0984693	11.23206	Exact
0.9	0.988134	–	–	–	–	1 R-R
	0.987566	1.906357	–	–	–	2
	0.986722	1.90551	5.029357	–	–	3
	0.986722	1.904472	5.024758	8.229732	–	4
	0.986674	1.904470	5.002739	8.218391	11.648586	5
	0.986674	1.904444	5.002583	8.100413	11.624439	6
	0.986665	1.904444	5.002046	8.099382	11.254577	7
	0.986659	1.904442	5.001975	8.095732	11.231025	Exact

In Table 1 the p_1 parameters are also reported, as obtained in [14] by means of the optimized Rayleigh quotient approach. It is immediately seen that these values do not give upper bounds to the true results, probably because of numerical instabilities and truncation errors.

Table 3. As Table 2 for $d = 0.4$.

k	p_1	p_2	p_3	p_4	p_5	N	
0.3	1.027437	–	–	–	–	1	R-R
	1.017921	2.743763	–	–	–	2	
	1.016667	2.376756	5.411609	–	–	3	
	1.016659	2.732156	5.410509	8.483562	–	4	
	1.016659	2.732156	5.410509	8.483562	11.847336	5	
	1.016608	2.732054	5.374444	8.334514	11.843373	6	
	1.016598	2.732051	5.373602	8.334261	11.422235	7	
	1.016593	2.732043	5.373524	8.329403	11.397314	Exact	
0.6	0.965018	–	–	–	–	1	R-R
	0.961179	2.288277	–	–	–	2	
	0.960136	2.288229	5.148701	–	–	3	
	0.960134	2.286193	5.147198	8.301470	–	4	
	0.960086	2.286193	5.121222	8.294988	11.702219	5	
	0.960086	2.286152	5.121123	8.168381	11.685179	6	
	0.960077	2.286152	5.120502	8.160502	11.301168	7	
	0.960072	2.286148	5.120429	8.163638	11.27734	Exact	
0.8	0.916352	–	–	–	–	1	R-R
	0.914836	2.125093	–	–	–	2	
	0.913937	2.124980	5.095878	–	–	3	
	0.913937	2.123424	5.093202	–	–	4	
	0.913892	2.123424	5.068980	8.261595	11.678775	5	
	0.913892	2.123390	5.068855	8.138749	11.658780	6	
	0.913884	2.123390	5.068272	8.137864	11.280903	7	
	0.913879	2.123387	5.068200	8.134034	11.257202	Exact	
1	0.868307	–	–	–	–	1	R-R
	0.867902	2.023784	–	–	–	2	
	0.867126	2.023395	5.069741	–	–	3	
	0.867126	2.022068	5.066382	8.255018	–	4	
	0.867084	2.022066	5.042996	8.245543	11.667713	5	
	0.867084	2.022037	5.042858	8.124484	11.646244	6	
	0.867076	2.022037	5.042293	8.123546	11.271272	7	
	0.867071	2.022034	5.042221	8.119783	11.247627	Exact	

For a taper ratio $\alpha = \beta = 1.4$ and for $N = 7$ the influence of various control parameters is reported in Table 4. In all the examined cases, the agreement between the exact and the approximate results seems to be excellent.

Table 4. As Table 2 for $\alpha = \zeta = 1.4$ and various k and d.

d	k	p_1	p_2	p_3	p_4	p_5	
0.4	0.3	1.099224	2.871888	5.628017	8.751734	12.005972	E
		1.099260	2.872076	5.628337	8.753567	12.100708	R-R
	0.6	1.025908	2.385761	5.392442	8.609704	11.906935	E
		1.025948	2.385894	5.392678	8.611277	12.001050	R-R
	0.9	0.940385	2.169849	5.334875	8.578956	11.886483	E
		0.940427	2.169955	5.335096	8.580495	11.980422	R-R
0.8	0.3	0.924807	2.933698	5.841787	8.970905	12.202877	E
		0.924844	2.933871	5.842214	8.973555	12.298625	R-R
	0.6	0.890276	2.576993	5.529228	8.708361	11.982542	E
		0.890314	2.577131	5.529507	8.710101	12.077407	R-R
	0.9	0.843940	2.339327	5.410326	8.628611	11.923145	E
		0.843979	2.339442	5.410566	8.630212	12.017500	R-R

Table 5. Approximate nondimensional frequency for $\alpha = 1.4$, $\zeta = 1.2$, $\beta = 0.5$ and $C_R = 0$.

μ	d	k	p_1	p_2	p_3	p_4	p_5
0.5	0.2	0.3	1.368165	3.056673	5.634578	8.661169	11.898079
		0.6	1.255789	2.534250	5.423743	8.546146	11.825337
		0.9	1.132205	2.331193	5.376987	8.522786	11.810998
	0.4	0.3	1.240350	3.056740	5.776883	8.814624	12.030156
		0.6	1.165143	2.633240	5.499110	8.603660	11.868796
		0.9	1.074593	2.412816	5.415877	8.550441	11.831360
	0.8	0.3	1.053822	2.993444	5.900715	9.017613	12.250954
		0.6	1.016910	2.743714	5.644804	8.744598	11.989575
		0.9	0.966730	2.544843	5.511208	8.630452	11.895516
	1	0.3	0.984939	2.963054	5.922323	9.069802	12.318963
		0.6	0.957829	2.770116	5.701402	8.811055	12.053157
		0.9	0.919496	2.592868	5.559351	8.676551	11.935121
1	0.2	0.3	1.168939	2.760406	5.439289	8.523803	11.797874
		0.6	1.066682	2.241971	5.296498	8.453499	11.755547
		0.9	0.957751	2.058088	5.267303	8.439846	11.747456
	0.4	0.3	1.053529	2.848604	5.581841	8.641322	11.887599
		0.6	0.986895	2.366765	5.351430	8.490151	11.781501
		0.9	0.907999	2.146974	5.293821	8.456826	11.759310
	0.8	0.3	0.890364	2.889795	5.769784	8.859918	12.089404
		0.6	0.858515	2.544490	5.480780	8.594130	11.862261
		0.9	0.815443	2.309825	5.366563	8.509907	11.798860
	1	0.3	0.831128	2.886524	5.819776	8.936130	12.172124
		0.6	0.807892	2.601927	5.542369	8.652038	11.911000
		0.9	0.775138	2.377581	5.408107	8.543412	11.825059

Finally, in Table 5 the free vibration frequencies of a cantilever beam for which $\alpha \neq \zeta$ and $\zeta \neq 1$ are given, as obtained with the Rayleigh - Ritz method, whereas the exact solution does not seem to be available.

5. Conclusions

The dynamic behaviour of a cantilever beam with continuously varying cross-section and with a tip mass has been studied both in an exact way and by means of the Rayleigh - Ritz approximation method. In the first case the solution of the differential equation of motion has been expressed in terms of Bessel functions, whereas in the second case a set of orthogonal polynomials has been used.

The numerical results show a good agreement between the exact values and the approximate upper bounds, and this is a very useful property for all the cases in which an exact solution is not attainable.

Appendix. Term of the matrix A, Eq. (2.18)

$$a_{11} = 1 - \mu Z d p^2, \qquad a_{12} = \mu Z(d^2 + k^2)p^3, \qquad a_{13} = -1 - \mu Z d p^2,$$
$$a_{14} = a_{12}, \qquad a_{15} = a_{16} = a_{17} = a_{18} = 0,$$

$$a_{21} = -\mu Z p, \qquad a_{22} = 1 + \mu Z p, \qquad a_{23} = -\mu Z p,$$
$$a_{24} = -1 + \mu Z d p^2, \qquad a_{25} = a_{26} = a_{27} = a_{28} = 0,$$

$$a_{31} = \cosh p\beta, \qquad a_{32} = \sinh p\beta, \qquad a_{33} = \cos p\beta,$$
$$a_{34} = \sin p\beta, \qquad a_{35} = -J_n(2q_a), \qquad a_{36} = -Y_n(2q_a),$$
$$a_{37} = -I_n(2q_a), \qquad a_{38} = -K_n(2q_a),$$

$$a_{41} = \sinh p\beta, \qquad a_{42} = \cosh p\beta, \qquad a_{43} = -\sin p\beta,$$
$$a_{44} = \cos p\beta, \qquad a_{45} = J_{n+1}(2q_a), \qquad a_{46} = Y_{n+1}(2q_a),$$
$$a_{47} = -I_{n+1}(2q_a), \qquad a_{48} = K_{n+1}(2q_a),$$

$$a_{51} = \cosh p, \qquad a_{52} = \sinh p\beta, \qquad a_{53} = -\cos p\beta,$$
$$a_{54} = -\sin p\beta, \qquad a_{55} = -J_{n+2}(2q_a), \qquad a_{56} = -Y_{n+2}(2q_a),$$
$$a_{57} = -I_{n+2}(2q_a), \qquad a_{58} = -K_{n+2}(2q_a),$$
$$a_{61} = p\sinh p\beta, \qquad a_{62} = p\cosh p\beta, \qquad a_{63} = p\sin p\beta,$$

$$a_{64} = -p\sin p\beta, \qquad a_{65} = pJ_{n+3}(2q_a) - (n+2)\frac{\alpha-1}{1-\beta}J_{n+2}(2q_a),$$

$$a_{66} = pY_{n+3}(2q_a) - (n+2)\frac{\alpha-1}{1-\beta}Y_{n+2}(2q_a),$$

$$a_{67} = -pI_{n+3}(2q_a) - (n+2)\frac{\alpha-1}{1-\beta}I_{n+2}(2q_a),$$

$$a_{68} = pK_{n+3}(2q_a) - (n+2)\frac{\alpha-1}{1-\beta}K_{n+2}(2q_a),$$

$$a_{75} = pC_R J_{n+2}(aa) - \alpha^{0.5}J_{n+1}(aa),$$

$$a_{76} = pC_R Y_{n+2}(aa) - \alpha^{0.5}Y_{n+1}(aa),$$

$$a_{77} = pC_R I_{n+2}(aa) + \alpha^{0.5}I_{n+1}(aa),$$

$$a_{78} = pC_R K_{n+2}(aa) - \alpha^{0.5}K_{n+1}(aa),$$

$$a_{85} = \alpha^2 J_n(aa), \qquad a_{86} = \alpha^2 Y_n(aa),$$

$$a_{87} = \alpha^2 I_n(aa), \qquad a_{88} = \alpha^2 K_n(aa),$$

with $aa = 2q_a a^{0.5}$.

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