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A CASE OF REFLECTION OF SIMPLE WAVE FROM A CONTACT DISCONTINUITY

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An exact analytical solution is presented for the wave system describing a one-dimensional unsteady process of nonlinear reflection of an arbitrary simple wave from a contact discontinuity dividing two ideal perfect gases of constant values of adiabatic indices k and k^0 which equal 3, and an arbitrary $\gamma > 1$, respectively. We suppose that the incident simple wave propagates through the gas of adiabatic index k equal to 3. As an example, we investigate the initial stage of a one-dimensional process of expansion of condensed-phase products of detonation in a medium with counterpressure.

1. Introduction

The phenomenon of interaction of a simple wave with contact discontinuity is an important element of many gasdynamical problems considered from the point of view of one-dimensional inviscid compressible model. The mathematical statement of the problem describing the interaction process mentioned above has been given by TAUB [7]. He has also obtained the solution for the interaction region of an incident wave and a wave reflected from the contact discontinuity in the case when adiabatic indices k and k^0 of the media at both sides of the contact discontinuity were equal to

$$(1.1) \qquad k = k^0 = (2n+3)/(2n+1), \qquad n = 0,1,2,\dots .$$

Another particular case of the problem under consideration was analysed by SOZONENKO [5], also with the restriction (1.1).

In the present paper we formulate the problem (Sec. 2) and obtain its exact analytical solution (Sec. 3) for the whole wave system arising from the reflection process, provided that $k = 3$ and $k^0 = \gamma$ for an arbitrary $\gamma > 1$. Moreover, the value $k = 3$ corresponds to the region of flow where the incident simple wave propagates. Obviously, the case under consideration has not been described by the solutions obtained by TAUB [7] and SOZONENKO [5] except for the case when $\gamma = 3$.

As shown by LANDAU and STANYUKOVICH [4], the value of adiabatic index $k = 3$ is characteristic for the products of detonation of condensed-phase explosives.

Therefore the solution obtained in the present paper may be used in gasdynamic analysis of the initial stage of detonation of explosives mentioned above, considered from the point of view of one-dimensional gasdynamic theory of instantaneous detonation (STANYUKOVICH [6]).

The results of such investigation are given in Sec. 4 of the present paper as an example of application of the general relations obtained in Sec. 2.

2. General statement of the problem

Let the surface of contact discontinuity separate initially two regions *1* and *2* of a one-dimensional rectilinear flow of ideal compressible fluid considered as a perfect gas with specific heat constants c_p and c_v, its initial state involving constant pressure $p_1 = p_2$, constant velocity $u_1 = u_2$ and constant density ϱ_1, ϱ_2 ($\varrho_1 \neq \varrho_2$, in general). In our discussion we may assume, without any loss of generality, that adiabatic indices of fluids from the left (k) and the right-hand side (k^0) of contact discontinuity are equal to $k = 3$ and $k^0 = \gamma$, with arbitrary $\gamma > 1$.

The equations of motion defining the one-dimensional, inviscid perfect gas flow in each of the half-spaces mentioned above, are then (STANYUKOVICH [6]):

$$
\begin{aligned}
&c\frac{\partial u}{\partial x} + \frac{2}{\gamma - 1}\left(\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x}\right) = 0,\\
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{2}{\gamma - 1}c\frac{\partial c}{\partial x} = 0,
\end{aligned}
\tag{2.1}
$$

where t, x are the time and space variables ([1]), respectively, u is the fluid velocity, c is the local sound speed and the ratio of specific heats γ is equal to 3 at the left-hand side of the contact discontinuity.

We shall assume that the wave interaction process under consideration is an isentropic process. Therefore we shall consider an arbitrary simple rarefaction wave ([2]) propagating in the positive x-axis direction through the

([1]) We shall assume below that the x-axis direction is the direction from the left to the right.

([2]) The case of a compressive incident wave can be investigated in the same manner. Strictly speaking, in this case equations (2.1) are not valid after the shock forms. The same restrictions take also place in the considered case of a rarefaction incident wave, if the wave reflected from the contact discontinuity is a compressive wave. Therefore we shall consider a strictly isentropic case when both the incident and reflected waves are rarefaction waves.

region of flow where adiabatic index k equals 3. The source of this wave (a moving piston or a local initial disturbance of flow) makes no difference for us, in principle. According to LANDAU and LIFSHITZ [3], the solution for the incident wave mentioned above is given by the expressions

$$
\begin{aligned}
x &= (u+c)t + f^*(u), \\
c - u &\equiv \text{const}\,,
\end{aligned}
\tag{2.2}
$$

where $f^*(u)$ is the known function of u. In the last identity it has been assumed that the value of adiabatic index k is equal to 3.

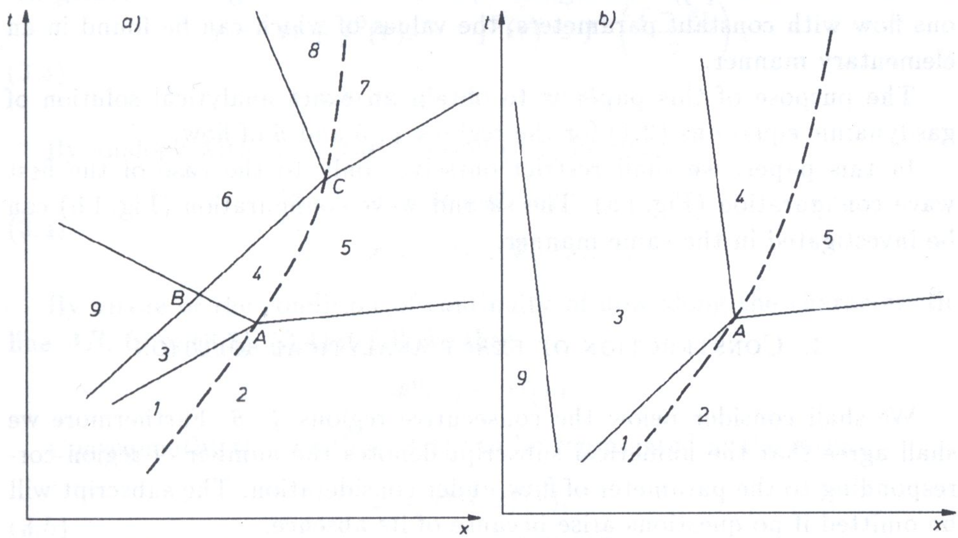

FIG. 1. Scheme of the wave reflection phenomenon; a) case $r_B > 0$, b) case $r_B \leq 0$.

The scheme of possible case of the phenomenon being investigated is represented in Fig. 1 a in the distance-time x, t-plane. Continuous lines denote the characteristic curves, dashed line is the trajectory of contact discontinuity. Let $x = X(t)$ be the equation of this line in x, t-plane. Numbers in figures denote the parts of the distance-time plane corresponding to different regions of flow under consideration.

In the course of the interaction of incident wave *3* with the contact discontinuity, the reflected (region *6*) and transmitted (region *5*) simple waves arise. The triangle ABC (region *4*) is the region of nonlinear interaction of incident and reflected waves.

As is well known (CHERNYI [1]), the incident simple wave is refracted on the contact discontinuity without changing the type. Therefore the wave *5* is a rarefaction wave. The reflected wave *6* may be both a rarefaction wave and a compression wave, depending on the initial data on contact discontinuity [3].

The scheme of the reflection phenomenon described above will occur provided the slope of the "tail" characteristic line of the incident wave *3* is positive. In the opposite case the process of nonlinear interaction of incident and reflected waves (the region *4*) is unlimited in time, and region *6* of the reflected simple wave is absent. This case corresponds to the Fig. 1 b.

Analytically, simple waves *5*, *6* and, on the other hand, the flow in region *4* are given by Riemann's solution and the compound wave solution of one-dimensional isentropic gasdynamic equations (2.1), respectively (LANDAU and LIFSHITZ [3]). The regions *7*, *8* and *9* are the regions of homogeneous flow with constant parameters, the values of which can be found in an elementary manner.

The purpose of this paper is to obtain an exact analytical solution of gasdynamic equations (2.1) for the regions *4*, *5* and *6* of flow.

In this paper, we shall restrict ourselves only to the case of the first wave configuration (Fig. 1 a). The second wave configuration (Fig. 1 b) can be investigated in the same manner.

3. Construction of exact analytical solution

We shall consider below the consecutive regions *4*–*6*. Furthermore we shall agree that the numerical subscript denotes the number of region corresponding to the parameter of flow under consideration. The subscript will be omitted if no questions arise because of its absence.

3.1. Region 4 of nonlinear wave interaction

According to STANYUKOVICH [6], for $\gamma = 3$, the general solution of Eqs. (2.1) which describe the compound wave in region *4* is of the form

$$\begin{aligned} x &= (u+c)t + \Psi^0(u+c), \\ x &= (u-c)t + \Psi(u-c), \end{aligned} \tag{3.1}$$

where Ψ^0, Ψ are arbitrary functions of their arguments. The function Ψ^0 can be found by means of conditions of compatibility of compound wave (3.1) and incident wave (2.2) along the characteristic line AB.

[3] We have already noticed that only the first case will be considered in detail (see footnote 2). The second case can be investigated in the same manner.

Let us introduce Riemann's invariants for regions to the left (r, s) and to the right (r^0, s^0) of the contact discontinuity, taking into account that the value of adiabatic index equals 3 in the half-space on one side of contact discontinuity ([4]):

$$r = c + u, \qquad s = c - u,$$

(3.2)

$$r^0 = \frac{2c}{\gamma - 1} + u, \qquad s^0 = \frac{2c}{\gamma - 1} - u.$$

In terms of new variables, the relations (2.2) for incident simple wave *3* can be represented in the form:

$$x = rt + f^0(r), \qquad f^0(r) \equiv f^*\left(\frac{r - s_1}{2}\right),$$

(3.3)

$$s = s_1 = c_1 - u_1 \equiv \text{const}.$$

By analogy with (2.2), the solution (3.1) becomes

(3.4)

$$x = rt + \Psi^0(r),$$
$$x = -st + \Psi(-s).$$

By virtue of the condition of continuity of flow along the characteristic line AB, from (3.3), (3.4) it follows that

$$\Psi^0(r) = f^0(r).$$

Consequently, the solution (3.4) can be represented in the form:

(3.5)

$$x = rt + 2f(r),$$
$$x = -st + 2F(s),$$

where $f(r) = (f^0(r))/2$ is the known function and $F(s) = (\Psi(-s))/2$ is the function to be found.

The unknown function can be found from the conditions of compatibility of the transmitted simple wave *5* with the solution (3.5) along the contact discontinuity line (curve AC). These conditions are the continuity conditions of flow velocity and gas pressure along the contact discontinuity line. They can be represented in terms of characteristic variables in the form

$$r - s = r^0 - s^0,$$

(3.6)

$$\left(\frac{r + s}{2c_1}\right)^3 = \left(\frac{(\gamma - 1)(r^0 + s^0)}{4c_2}\right)^{2\gamma/(\gamma - 1)}.$$

([4]) Above we have already assumed that $k = 3$ at the left side of the contact discontinuity.

If it is remembered that invariant s is constant everywhere ($s^0 \equiv s_2^0$) in the region *5* of the transmitted wave, then we obtain the resultant relation between r and s on the contact discontinuity by excluding r^0 from (3.6):

$$(3.7) \qquad r + s = 2c_1 \left[\frac{\gamma - 1}{4c_2} \left(r - s + 2s_2^0 \right) \right]^{1-\chi}, \qquad \chi = \frac{\gamma - 3}{3(\gamma - 1)}.$$

Let us transform Eqs. (3.5). Subtracting the second equality from the first one and taking into account the notations (3.2), we obtain

$$(3.8) \qquad \begin{gathered} ct + f(r) - F(s) = 0, \\ t = \frac{F(s) - f(r)}{c} = 2\frac{F(s) - f(r)}{r + s}. \end{gathered}$$

Analogously, adding both Eqs. (3.5), we can find:

$$(3.9) \qquad x = ut + f(r) + F(s).$$

The relations (3.8), (3.9) are valid everywhere in region *4* of the nonlinear wave interaction.

By differentiating Eq. (3.9) along the contact discontinuity (the curve AC) where $dX = U\,dt$, with U denoting the velocity of the contact discontinuity, we obtain

$$dX = U\,dt + t\,dU + d[f(r) + F(s)],$$

so that

$$(3.10) \qquad t\,dU + d[f(r) + F(s)] = 0.$$

Taking into account the relations (3.8) and the continuity of flow velocity on the contact discontinuity (cd)

$$U = u\Big|_{cd} = \frac{r - s}{2}\Big|_{cd},$$

we obtain

$$(3.11) \qquad [F(s) - f(r)]d(r - s) + (r + s)d[F(s) + f(r)] = 0.$$

In Eq. (3.11) Riemann's invariants r and s satisfy relation (3.7) on the contact discontinuity.

Equation (3.11) together with Eq. (3.7) are considered as a differential equation enabling the determination of the unknown function $F(s)$.

To solve this equation, it is convenient to introduce the following substitution:

$$w = r - s + s_2^0 .$$

Hence

$$r - s = w - 2s_2^0 ,$$
$$r + s = 2c_1\left(\frac{\gamma - 1}{2c_2}w\right)^{1-\chi},$$

and consequently

$$(3.11') \qquad \begin{aligned} r &= c_1\left(\frac{\gamma-1}{4c_2}w\right)^{1-\chi} + \frac{w}{2} - s_2^0 , \\ s &= c_1\left(\frac{\gamma-1}{4c_2}w\right)^{1-\chi} - \frac{w}{2} + s_2^0 . \end{aligned}$$

Denote also

$$(3.12) \qquad Z(w) = F(s(w)) + f(r(w)),$$

where $f(r(w)) \equiv q(w)$ is a known function of w.

Finally, the system of Eqs. (3.11), (3.7) reduces to one equation

$$(3.13) \qquad \frac{dZ}{dw} + P(w)Z = Q(w),$$

where

$$P(w) = \frac{1}{2c_1}\left(\frac{\gamma-1}{4c_2}w\right)^{\chi-1}, \qquad Q(w) = \frac{q(w)}{c_1}\left(\frac{\gamma-1}{4c_2}w\right)^{\chi-1}.$$

Equation (3.13) is a first order linear nonhomogeneous ordinary differential equation. Its solution passing through the point (ξ, η) can be represented in a closed form by quadratures:

$$(3.14) \qquad Z = e^{-Y}\left(\eta + \int_{\xi}^{w} Q(w)e^{Y}\,dw\right), \qquad Y = \int_{\xi}^{w} P(w)\,dw.$$

In the case under consideration

$$(3.14') \qquad \begin{aligned} \xi &= w\Big|_A = r_1 - s_1 + 2s_2^0 = \frac{4c_2}{\gamma - 1}, \\ \eta &= Z\Big|_A = (F + f)\Big|_A = (x - ut)\Big|_A . \end{aligned}$$

Calculation of $Y(w)$ according to (3.13) gives for $\gamma \neq 3$ $(\chi \neq 0)$

$$Y = \int_{\xi}^{w} P(w)\,dw = \frac{1}{2c_1}\int_{\xi}^{w}\left(\frac{\gamma-1}{4c_2}w\right)^{\chi-1} dw = \frac{6c_2}{(\gamma-1)c_1}\left[\left(\frac{\gamma-1}{4c_2}w\right)^{\chi} - 1\right]. \tag{3.15}$$

For $\gamma = 3$ $(\chi = 0)$ Eq. (3.13) becomes

$$\frac{dZ}{dw} + \frac{c_2}{c_1}\frac{Z}{w} - 2\frac{c_2}{c_1}\frac{q(w)}{w} = 0.$$

Its solution, according to (3.14), is

$$Z = (w/\xi)^{-(c_2/c_1)}\left[\eta + \frac{1}{c_1}\int_{\xi}^{w} q(w)(w/\xi)^{c_2/c_1-1}\,dw\right], \qquad \xi = \frac{4c_2}{\gamma-1} = 2c_2\,. \tag{3.16}$$

Then the unknown function $F(s)$ is defined by the relation

$$F(w) = Z(w) - q(w), \tag{3.17}$$

where $Z(w)$ for $\gamma \neq 3$ is defined by Eqs. (3.14), (3.15), and for $\gamma = 3$ by Eqs. (3.16). Function $q(w)$ is the known function generated by the incident simple wave. Function $w(s)$ is given implicitly by Eqs. (3.11′).

3.2. Region 6 of reflected simple wave

Solution in region *6* for $k = 3$ has the following form

$$x = -st + \Psi(s), \qquad r = r_9 \equiv \text{const}\,, \tag{3.18}$$

where $\Psi(s)$ is an unknown function.

In the case under consideration the function can be easily found if we use the well-known (STANYUKOVICH [6]) property of independence of direct and inverse wave propagation during the process of their nonlinear interaction under the condition $k = 3$.

It is evident that

$$\Psi(s) = 2F(s), \tag{3.18′}$$

where $F(s)$ has been found above in Sec. 3.1.

3.3. Region 5 of transmitted simple wave

The solution describing simple wave 5 is of the form

$$x = (u + c)t + G(w^0), \qquad w^0 = r^0 - s^0 + 2s_2^0, \tag{3.19}$$
$$s^0 = s_2^0 \equiv \mathrm{const}$$

with a function $G(w^0)$ which must be found.

We note that according to (3.6)

$$w = w^0 \tag{3.20}$$

on the contact discontinuity.

Differentiating (3.19) along the curve AC we find, by analogy with (3.9), that

$$dG(w^0) + c_5\,dt + t\,d(U + c_5) = 0 \tag{3.21}$$

or using (3.10), (3.12) and (3.20)

$$d\left[G(w^0) + c_5 t\right] = dZ(w^0). \tag{3.22}$$

Integrating (3.22) we find

$$G(w^0) = Z(w^0) - c_5 t. \tag{3.23}$$

Moreover, according to (3.8), (3.10) and (3.20)

$$t = \frac{F(s) - f(r)}{c_4} = \frac{Z(w^0) - 2q(w^0)}{c_4}, \tag{3.24}$$

and the second equality (3.6) gives the relation between the sound velocities on part AC of the contact discontinuity trajectory in the form

$$c_4 = c_1 \left(\frac{c_5}{c_2}\right)^{1-\chi}. \tag{3.25}$$

Substituting Eqs. (3.24), (3.25) into (3.23) we finally obtain

$$G(w^0) = \left[1 - \frac{c_2}{c_1}\left(\frac{c_5(w^0)}{c_2}\right)^{\chi}\right] Z(w^0) + 2\frac{c_2}{c_1}\left[\frac{c_5(w^0)}{c_2}\right]^{\chi} q(w^0), \tag{3.26}$$

where

$$c_5(w^0) = \frac{\gamma - 1}{4} w^0.$$

3.4. The contact discontinuity trajectory

To obtain a complete solution of the problem under consideration it still remains to investigate the law governing the contact discontinuity motion from point A to point C, because the motion outside AC is a motion with constant velocities U_A, U_C which can be found in an elementary manner.

To obtain the function $X(t)$ which defines part AC of the desired trajectory, let us make use of the relation (3.9). This relation is valid everywhere in region *4* up to its boundary AC. According to (3.9) we have for arbitrary point t, $X(t)$ of curve AC

$$X = Ut + \zeta(U), \tag{3.27}$$

where

$$U = dX/dt, \qquad \zeta(U) \equiv Z(2U + 2s_2^0).$$

Equations (3.27) represent the differential equations of contact discontinuity trajectory.

Since Eq. (3.27) is of Clairaut type, it can always be solved in a closed form.

Thus, the singular solution interesting for us is written in the following parametric form (KAMKE [2]):

$$t = -V'(U), \qquad X = -UV'(U) + V(U). \tag{3.28}$$

In view of relations (3.14′), the solution (3.28) certainly satisfies the required initial condition

$$x_A = X(t_A).$$

4. EXAMPLE. THE INITIAL STAGE OF ONE-DIMENSIONAL EXPANSION PHENOMENON OF CONDENSED-PHASE DETONATION PRODUCTS

We now consider a well-known (STANYUKOVICH [6]) problem of one-dimensional expansion of plane compressed gas layer (initially at rest) into a medium with counterpressure, as an example of using the formulae which have been obtained in Sec. 3. We suppose that adiabatic index k in the layer equals 3.

As shown by LANDAU and STANYUKOVICH [4], the value $k = 3$ is typical for products of condensed-phase explosions. Therefore our solution which will be derived below represents an exact gasdynamic description of the initial stage of detonation of such explosives, investigated from the point of view of the one-dimensional instantaneous detonation model. It allows us to extend the exact analytical description of expansion of the explosion products to the time period later than the well-known (STANYUKOVICH [6]) solution. As in the general case which was analysed in Sec. 3, here it is possible to encounter two situations represented in Figs. 2 a, 2 b. The origin of coordinates coincides with the left-hand boundary of layer.

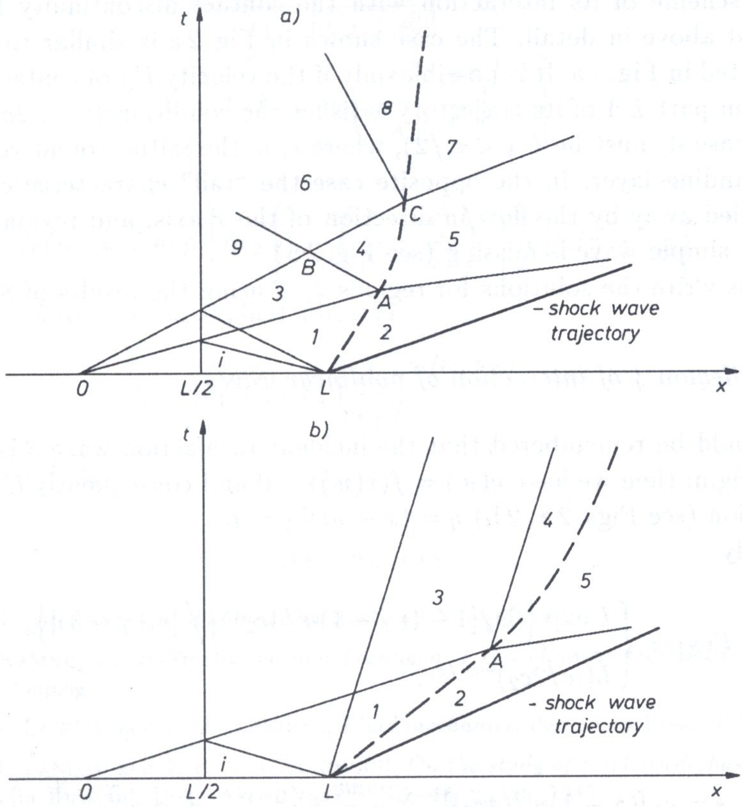

FIG. 2. Scheme of the initial stage of one-dimensional expansion of condensed-phase detonation products; a) case $U_A < c_i/2$, b) case $U_B > c_i/2$.

Evidently, the flow under consideration is symmetric with respect to the plane $x = L/2$, where L is the thickness of the expanding, compressed layer. Therefore we shall restrict ourselves to the examination of only the right-hand ($x > L/2$) half-plane of the x, t-plane.

Initially, the boundary of the layer is an arbitrary discontinuity surface which disintegrates on a rarefaction wave *2* centred at point $(L, 0)$, running to the centre of symmetry, the contact discontinuity surface separating the detonation products from the external medium, and a shock wave propagating away from the initial explosion.

The simple wave *3* (incident wave in Sec. 3) is a result of reflection of the centred wave *2* from the plane of symmetry $x = L/2$. Therefore it is a rarefaction wave too (LANDAU and LIFSHITZ [3]). Since for $k = 3$ the characteristics of one-dimensional gasdynamic equations (2.1) are always rectilinear (LANDAU and LIFSHITZ [3], STANYUKOVICH [6]), the wave *3* (by virtue of the symmetry) turns out to be centred around the origin.

The scheme of its interaction with the contact discontinuity has been discussed above in detail. The case shown in Fig. 2 a is similar to the case represented in Fig. 1 a. It is possible only if the velocity U_A of contact discontinuity on part LA of its trajectory satisfies the condition $U_A < 2c_i/(k+1)$ (in our case it must be $U_A < c_i/2$), where c_i is the initial sound velocity in the expanding layer. In the opposite case the "tail" characteristic of wave *2* is carried away by the flow in direction of the x-axis, and region *6* of the reflected simple wave is missing (see Fig. 2 b).

Let us write the solutions for regions *4*–*6* using the results of Sec. 3.

4.1. Region 4 of interaction of nonlinear waves

If should be remembered that the incident rarefaction wave *3* is centred in the origin; then we have $q(w) = f(r(w)) \equiv 0$ and consequently $Q(w) \equiv 0$. In addition (see Figs. 2 a, 2 b) $\eta = (x - ut)|_A = L$.

Finally

$$(4.1) \qquad F(s) = \begin{cases} L\exp\left\{6c_2\left[1 - ((\gamma-1)w/4c_2)^{\chi}\right] \big/ \left[c_1(\gamma-3)\right]\right\}, & \gamma \neq 3, \\ L(w/2c_2)^{-c_2/c_1}, & \gamma = 3, \end{cases}$$

where

$$s = c_1\left[(\gamma-1)w/(4c_2)\right]^{1-\chi} - \frac{w}{2} + s_2^0\,, \qquad \chi = \frac{\gamma-3}{3(\gamma-1)}\,.$$

4.2. Region 6 of the reflected wave

The assumption given by formula (3.18′) is applied here to the relations (4.1).

4.3. Region 5 of the transmitted wave

In solution (3.26) it is necessary to put $q(w) \equiv 0$ and $z \equiv F$, function F being defined according to (4.1).

4.4. The trajectory of contact discontinuity

For $\gamma \neq 3$ the parametric representation (3.28) of contact discontinuity line in x, t-plane in view of (4.1) is of the form

$$t = \frac{\zeta_0(U)}{c_1}\left[\frac{U+s_2^0}{U_A+s_2^0}\right]^{\chi-1},$$

$$X = t\left[U + c_1\left(\frac{U+s_2^0}{U_A+s_2^0}\right)^{1-\chi}\right],$$

where

$$\zeta_0(U) = L\exp\left\{6c_2\left[1-\left(\frac{U+s_2^0}{U_A+s_2^0}\right)^{\chi}\right]\Big/[c_1(\gamma-1)]\right\}.$$

For $\gamma = 3$ we find an explicit formula

$$X = (c_1+c_2)t_A\left(\frac{t}{t_A}\right)^{c_2/(c_1+c_2)} - s_2^0 t,$$

where $T_A = L/c_1$.

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