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NEW INTERPOLATION FUNCTIONS IN EIGEN-FREQUENCY ANALYSIS OF TIMOSHENKO BEAMS ON TWO-PARAMETER ELASTIC SOIL

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The aim of the paper is to study the dynamic behaviour, limited to the eigen-value problem, of Timoshenko beams on variable two-parameter elastic soil. The analysis is performed by means of two different finite elements, with cubic and quintic interpolation laws, the mass and stiffness matrices are analytically calculated, and their performances are briefly discussed. Some numerical examples end the paper, in which the good convergence rate of the elements is shown, and a comparison is made with a powerful Rayleigh–Ritz approximation.

1. Introduction

The simplest structural model of a foundation beam is given by an Euler - Bernoulli slender beam resting on a Winkler elastic soil. The classical HETENYI results for the static analysis [1] refer to this model, and endless finite elements have been proposed, in order to study stability and dynamic behaviours of the beam. For example, the exact analyses given in [2, 3] must be noted, together with an attempt to analyze beams on variable Winkler soil [4].

Nevertheless, this simple model has been questioned in two respects.

Firstly, it is well-known that the Euler - Bernoulli hypothesis can be accepted only for slender beams, in which the shear deformations can be neglected, and usually a foundation beam is rather stocky. Moreover, the higher vibration modes are always affected by significant shear effects.

A substantial improvement can be obtained, if the so-called Timoshenko theory is employed, in which the shear energy is taken into account in a simplified model, by introducing a corrective factor. The strain energy of the beam is therefore written as:

$$S = S_b + S_s = \frac{1}{2}\int_0^L EIv''^2 dz + \frac{1}{2}\int_0^L GA\kappa\psi^2 dz, \tag{1.1}$$

where L is the span of the beam, E is the Young modulus, I is the second moment of area of the beam cross-section, $v(z,t)$ is the vertical displacement, G is the shear modulus, A is the cross-sectional area, κ is the corrective shear factor, and ψ is the additional shear angle. Finally, the primes denote derivatives with respect to the abscissa z.

If the Timoshenko beam theory is adopted, then the beam is supposed to be stocky, and therefore it is necessary to consider even the effects of the rotatory inertia of the cross-section. The kinetic energy is therefore given by:

$$T = T_t + T_r = \frac{1}{2}\int_0^L \varrho A \dot{v}^2 dz + \frac{1}{2}\int_0^L \varrho I \dot{\phi}^2 dz, \tag{1.2}$$

where ϱ is the mass density, $\phi = v' + \psi$ is the section rotation and the dot denotes derivatives with respect to time t.

Despite its great simplicity, even the Winkler soil model has been subjected to severe criticisms, a least in the presence of concentrated loads and flexible soil. Moreover, the Winkler soil model becomes unrealistic if higher vibration modes must be calculated.

A more accurate soil model goes back to Vlasov, where the soil is regarded as an elastic medium defined by the Young modulus E_s and the Poisson ratio ν_s. By means of simplifying hypotheses, the strain energy of the soil can be written as:

$$S_w = \frac{1}{2}\int_0^L k_w v^2 + \frac{1}{2}\int_0^L k_p \phi^2 dz, \tag{1.3}$$

where the soil parameters k_w and k_p can be expressed as functions of E_s and ν_s. More precisely, let us suppose that the Young modulus increases linearly from E_1 at the soil level to E_2 at the depth H:

$$E_s(z) = E_1\left(1 - \frac{z}{H}\right) + E_2\frac{z}{H}. \tag{1.4}$$

Then it will be [5]:

$$k_w = \frac{B(1-\nu_s)}{8H(1+\nu_s)(1-2\nu_s)} \times \left[\frac{E_1(2\gamma\sinh 2\gamma + 4\gamma^2) + (E_2 - E_1)(\cosh 2\gamma - 1 + 2\gamma^2)}{\sinh^2\gamma}\right], \tag{1.5}$$

$$(1.6)\qquad k_p = \frac{BH}{32\gamma^2(1+\nu_s)} \times \left[\frac{E_1(2\gamma\sinh 2\gamma + 4\gamma^2) + (E_2 - E_1)(\cosh 2\gamma - 1 + 2\gamma^2)}{\sinh^2\gamma}\right],$$

where B is the foundation width and γ is a parameter which can define the soil behaviour. It is worth noting that γ is influenced by the loading, so that its evaluation in dynamic analyses can be difficult.

2. The finite elements

The simplest finite element which can be used is the bilinear element, where both $v(z)$ and $\phi(z)$ are assumed to vary along the element according to a linear law:

$$(2.1)\qquad v(z) = A_0 + A_1 z, \qquad \phi(z) = A_2 + A_3 z.$$

Unfortunately, it is known that this element is subjected to severe locking phenomena [6], and it must be modified using, for example, the Prathap field consistency theory.

Better results are obtained by employing quadratic Mindlin elements or the so-called TIM7 finite elements [7].

In this paper we use two higher order elements, in which $v(z)$ and $\phi(z)$ are allowed to vary according to the cubic law [8 – 9]:

$$(2.2)\qquad v(z) = A_0 + A_1 z + A_2 z^2 + A_3 z^3, \qquad \phi(z) = A_4 + A_5 z + A_6 z^2 + A_7 z^3,$$

or with quintic law [10]:

$$(2.3)\qquad \begin{aligned} v(z) &= A_0 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4 + A_5 z^5,\\ \phi(z) &= A_6 + A_7 z + A_8 z^2 + A_9 z^3 + A_{10} z^4 + A_{11} z^5. \end{aligned}$$

In both the cases, the elements have two nodes, the degrees of freedom are given by:

$$(2.4)\qquad \mathbf{d}^T = (v_1, v_1', \phi_1, \phi_1', v_2, v_2', \phi_2, \phi_2')$$

for the cubic element, and:

$$(2.5)\qquad \mathbf{d}^T = (v_1, v_1', v_1'', \phi_1, \phi_1', \phi_1'', v_2, v_2', v_2'', \phi_2, \phi_2', \phi_2'')$$

for the quintic element.

The corresponding shape functions $N_i(z)$ are the usual Hermitian polynomials of degree 3 or 5, respectively, so that it is possible to write:

$$\mathbf{v} = \begin{pmatrix} v \\ \phi \end{pmatrix} = \mathbf{N}\mathbf{d}. \tag{2.6}$$

If this relationship is introduced into Eqs. (1.1)–(1.3), then after some algebra we can define the stiffness matrix of the structure:

$$\mathbf{k} = \int_0^L \mathbf{B}^T \mathbf{E} \mathbf{B}\, dz, \tag{2.7}$$

the stiffness matrix of the soil:

$$\mathbf{k}_s = \int_0^L \mathbf{N}^T \mathbf{W} \mathbf{N}\, dz, \tag{2.8}$$

and the mass matrix:

$$\mathbf{m} = \int_0^L \mathbf{N}^T \tilde{\mathbf{m}} \mathbf{N}\, dz, \tag{2.9}$$

where $\mathbf{B}$ is the *deformation matrix*, which can be obtained from the shape functions by means of differentiations, and the three diagonal matrices $\mathbf{E}$, $\mathbf{W}$ and $\tilde{\mathbf{m}}$ are given by:

$$\mathbf{E} = \begin{pmatrix} EI & 0 \\ 0 & GA\kappa \end{pmatrix}, \qquad \mathbf{W} = \begin{pmatrix} k_w & 0 \\ 0 & k_p \end{pmatrix}, \qquad \tilde{\mathbf{m}} = \begin{pmatrix} \varrho A & 0 \\ 0 & \varrho I \end{pmatrix}. \tag{2.10}$$

It is immediately seen that, if all the parameters are constant, then the stiffness matrix of the soil can be obtained from the mass matrix by identifying k_w with ϱA and k_p with ϱI. However, in the following, the soil parameters will be supposed to vary along the element according to a linear law, so that it will be:

$$k_w(z) = k_{wl} + (k_{wr} - k_{wl})\frac{z}{l}, \qquad k_p(z) = k_{pl} + (k_{pr} - k_{pl})\frac{z}{l}, \tag{2.11}$$

where k_{wl}, k_{wr}, k_{pl} and k_{pr} are the soil parameters at the left- and at the right-hand end, respectively.

In this case, of course, the soil stiffness matrix is no more deducible from the mass matrix, and is given in the Appendix both for the cubic and the quintic element, together with the stiffness matrix for the quintic element.

The mass matrices can be immediately recovered by putting $k_{wl} = k_{wr} \equiv \varrho A$ and $k_{pl} = k_{pr} \equiv \varrho I$.

The usual assembly procedure gives the global stiffness matrix **K** and global mass matrix **M**, and the following eigenvalue problem:

$$[-\omega^2 \mathbf{M} + \mathbf{K}]\mathbf{D} = 0 \tag{2.12}$$

must be solved, in order to deduce the free vibration frequencies ω_i^2 and the corresponding vibration modes.

3. Numerical results

Let us consider a simply supported beam defined by $E/G = 13/5$, $I/AL^2 = 0.04$ and $\kappa = 0.85$. Moreover, let us assume that the beam is resting on a linearly varying two-parametric elastic soil, where the soil coefficients are supposed to vary according to the following triangular laws:

$$k_w = K_w \frac{EI}{L^4}\frac{z}{L}, \qquad k_p = K_p \pi^2 \frac{EI}{L^2}\frac{z}{L}. \tag{3.1}$$

Table 1. First three nondimensional frequencies for different finite element discretization levels.

N_{elem}		1	2	3	4	5	10
Bicubic	Ω_1	3.73225	3.67114	3.66915	3.66892	3.66888	3.66886
	Ω_2	6.72073	6.24137	6.15496	6.14929	6.14800	6.14738
	Ω_3	19.0036	9.00990	8.64982	8.57983	8.57168	8.56749
Biquintic	Ω_1	3.66895	3.66886	3.66886	3.66886	3.66886	3.66886
	Ω_2	6.15747	6.14766	6.14737	6.14736	6.14736	6.14736
	Ω_3	9.27202	8.57087	8.56778	8.56738	8.56736	8.56736

In Table 1 the first three nondimensional frequencies:

$$\Omega_1 = \left(\frac{\varrho A L^4 \omega_i^2}{EI}\right)^{1/4} \tag{3.2}$$

are given, with $K_w = 100$ and $K_p = 1$, for increasing number of finite elements, and both for the bicubic and the biquintic element.

It is immediately seen that the convergence rate is quite good, even for the higher eigenvalues, expecially for the biquintic element.

In order to perform a comparison, a Rayleigh - Ritz approach has been employed, by approximating deflection and slope with the following formulae:

$$\text{(3.3)}\qquad \begin{aligned} v(z) &= A_1 \sin\frac{\pi z}{L} + A_2 \sin\frac{2\pi z}{L} + A_3 \sin\frac{3\pi z}{L}, \\ \phi(z) &= A_4 \cos\frac{\pi z}{L} + A_5 \cos\frac{2\pi z}{L} + A_6 \cos\frac{3\pi z}{L}. \end{aligned}$$

If the soil parameters variation law (3.1) is again adopted, then the resulting stiffness matrix is given by:

$$k_{1,1} = \frac{\pi^2}{L} GA\kappa + \frac{K_w EI}{2L^3}, \qquad k_{1,2} = \frac{-16 K_w EI}{9L^3\pi^2},$$
$$k_{1,3} = 0, \quad k_{1,4} = -GA\kappa\pi, \quad k_{1,5} = 0, \quad k_{1,6} = 0,$$
$$k_{2,2} = \frac{4\pi^2}{L} GA\kappa + \frac{K_w EI}{2L^3}, \quad k_{2,3} = \frac{-48 K_w EI}{25 L^3 \pi^2},$$
$$k_{2,4} = 0, \quad k_{2,5} = -2\pi GA\kappa, \quad k_{2,6} = 0,$$
$$k_{3,3} = \frac{18L^2\pi^2 GA\kappa + K_w EI}{2L^3}, \quad k_{3,4} = 0, \quad k_{3,5} = 0, \quad k_{3,6} = -3\pi GA\kappa,$$
$$k_{4,4} = GA\kappa L + \frac{\pi^2 EI}{L} + \frac{K_p \pi^2 EI}{2L}, \quad k_{4,5} = \frac{-20 K_p EI}{9L}, \quad k_{4,6} = 0,$$
$$k_{5,5} = GA\kappa L + \frac{4\pi^2 EI}{L} + \frac{K_p \pi^2 EI}{2L}, \quad k_{5,6} = \frac{-52 K_p EI}{25L},$$
$$k_{6,6} = \frac{2GA\kappa L^2 + 18\pi^2 EI + K_p \pi^2 EI}{2L}$$

and the mass matrix turns out to be diagonal, with non-zero coefficients given by:

$$\text{(3.4)}\qquad \begin{aligned} m_{1,1} &= m_{2,2} = m_{3,3} = \varrho AL, \\ m_{4,4} &= m_{5,5} = m_{6,6} = \varrho IL. \end{aligned}$$

In Table 2 the first three non-dimensional free frequencies are given, for the same beam as in Table 1, and for different values of K_w and K_p. The first row of each entry refers to the finite element results, as obtained with 15 biquintic elements, whereas the second row gives the Rayleigh - Ritz upper bounds.

The agreement seems to be quite good, even for the higher frequencies, at least if the soil is not too strong.

Table 2. First three nondimensional frequencies for different values of the soil parameters, as obtained with the quintic finite element (top rows) and with the Rayleigh–Ritz approach (bottom rows).

K_p	Ω_i	$K_w = 10$	$K_w = 100$	$K_w = 1000$	$K_w = 10^6$
0.5	Ω_1	3.29416 3.29476	3.56452 3.56500	4.89832 4.89856	18.95608 19.13355
	Ω_2	6.02837 6.02837	6.07732 6.07732	6.525300 6.525330	19.18400 19.58606
	Ω_3	8.50945 8.50962	8.52649 8.52676	8.692984 8.695600	19.79611 20.42891
1	Ω_1	3.42665 3.42870	3.66886 3.67053	4.93374 4.93450	18.99947 19.17865
	Ω_2	6.09993 6.09994	6.14736 6.14738	6.58279 6.58285	19.24783 19.63256
	Ω_3	8.55051 8.55114	8.56736 8.56816	8.73208 8.73587	19.84655 20.47982
2.5	Ω_1	3.72589 3.73494	3.91585 3.92367	5.02867 5.03249	19.09623 19.29801
	Ω_2	6.29034 6.29041	6.33396 6.33403	6.73733 6.73751	19.45441 19.78788
	Ω_3	8.66565 8.66910	8.68199 8.68578	8.84179 8.85020	20.00055 20.63504
10	Ω_1	4.45122 4.50784	4.56118 4.61406	5.35261 5.38666	19.38464 19.70479
	Ω_2	6.91599 6.91791	6.94906 6.95102	7.26129 7.26388	20.31278 20.54292
	Ω_3	9.10658 9.13866	9.12100 9.15389	9.26255 9.30414	20.83346 21.44102

4. Conclusion

The dynamic analysis of a Timoshenko beam resting on a variable two-parameter elastic soil has been performed, by using two finite elements with cubic and quintic interpolation law, respectively. The numerical examples show a high precision even if a small number of elements are used.

Appendix

The soil stiffness matrix of the cubic finite elements can be written as:

$$k_{11} = \frac{(10k_{wl} + 3k_{wr})L}{35}, \quad k_{12} = \frac{(15k_{wl} + 7k_{wr})L^2}{420}, \quad k_{13} = 0,$$

$$k_{14} = 0, \quad k_{15} = \frac{9(k_{wl} + k_{wr})L}{140}, \quad k_{16} = \frac{-(7k_{wl} + 6k_{wr})L^2}{420},$$

$$k_{17} = 0, \quad k_{18} = 0, \quad k_{22} = \frac{(5k_{wl} + 3k_{wr})L^3}{840},$$

$$k_{23} = 0, \quad k_{24} = 0, \quad k_{25} = \frac{(6k_{wl} + 7k_{wr})L^2}{420},$$

$$k_{26} = \frac{-(k_{wl} + k_{wr})L^3}{280}, \quad k_{27} = 0, \quad k_{28} = 0,$$

$$k_{33} = \frac{(10k_{pl} + 3k_{pr})L}{35}, \quad k_{34} = \frac{(15k_{pl} + 7k_{pr})L^2}{420}, \quad k_{35} = 0,$$

$$k_{36} = 0, \quad k_{37} = \frac{9(k_{pl} + k_{pr})L}{140}, \quad k_{38} = \frac{-(7k_{pl} + 6k_{pr})L^2}{420},$$

$$k_{44} = \frac{(5k_{pl} + 3k_{pr})L^3}{840}, \quad k_{45} = 0, \quad k_{46} = 0,$$

$$k_{47} = \frac{(6k_{pl} + 7k_{pr})L^2}{420}, \quad k_{48} = \frac{-(k_{pl} + k_{pr})L^3}{280},$$

$$k_{55} = \frac{(3k_{wl} + 10k_{wr})L}{35}, \quad k_{56} = \frac{-(7k_{wl} + 15k_{wr})L^2}{420},$$

$$k_{57} = 0, \quad k_{58} = 0, \quad k_{66} = \frac{(3k_{wl} + 5k_{wr})L^3}{840},$$

$$k_{67} = 0, \quad k_{68} = 0, \quad k_{77} = \frac{(3k_{pl} + 10k_{pr})L}{35},$$

$$k_{78} = \frac{-(7k_{pl} + 15k_{pr})L^2}{420}, \quad k_{88} = \frac{(3k_{pl} + 5k_{pr})L}{840}.$$

The stiffness matrix of the quintic element is given by:

$$k_{1,1} = \frac{10GA\kappa}{7L}, \quad k_{1,2} = \frac{3GA\kappa}{14}, \quad k_{1,3} = \frac{GA\kappa L}{84},$$

$$k_{1,4} = \frac{-GA\kappa}{2}, \quad k_{1,5} = \frac{-11GA\kappa L}{84}, \quad k_{1,6} = \frac{-GA\kappa L^2}{84},$$

$$k_{1,7} = -k_{1,1}, \quad k_{1,8} = k_{1,2}, \quad k_{1,9} = -k_{1,3}, \quad k_{1,10} = k_{1,4},$$

$$k_{1,11} = -k_{1,5}\,, \qquad k_{1,12} = k_{1,6}\,, \qquad k_{2,2} = \frac{8GA\kappa L}{35}\,,$$

$$k_{2,3} = \frac{GA\kappa L^2}{60}\,, \qquad k_{2,4} = k_{1,11}\,, \qquad k_{2,5} = 0,$$

$$k_{2,6} = \frac{-GA\kappa L^3}{1008}\,, \qquad k_{2,7} = -k_{1,2}\,, \qquad k_{2,8} = \frac{-GA\kappa L}{70}\,,$$

$$k_{2,9} = \frac{GA\kappa L^2}{210}\,, \qquad k_{2,10} = k_{1,5}\,, \qquad k_{2,11} = \frac{13GA\kappa L^2}{420}\,,$$

$$k_{2,12} = \frac{-13GA\kappa L^3}{5040}\,, \qquad k_{3,3} = \frac{GA\kappa L^3}{630}\,, \qquad k_{3,4} = -k_{1,6}\,, \qquad k_{3,5} = -k_{2,6}\,,$$

$$k_{3,6} = 0, \qquad k_{3,7} = -k_{1,3}\,, \qquad k_{3,8} = -k_{2,9}\,, \qquad k_{3,9} = \frac{GA\kappa L^3}{1260}\,,$$

$$k_{3,10} = k_{1,6}\,, \qquad k_{3,11} = -k_{2,12}\,, \qquad k_{3,12} = \frac{-GA\kappa L^4}{5040}\,,$$

$$k_{4,4} = \frac{10EI}{7L} + \frac{181GA\kappa L}{462}\,, \qquad k_{4,5} = \frac{3EI}{14} + \frac{311GA\kappa L^2}{4620}\,,$$

$$k_{4,6} = \frac{EIL}{84} + \frac{281GA\kappa L^3}{55440}\,, \qquad k_{4,7} = -k_{1,4}\,, \qquad k_{4,8} = k_{1,5}\,,$$

$$k_{4,9} = -k_{1,6}\,, \qquad k_{4,10} = \frac{-10EI}{7L} + \frac{25GA\kappa L}{231}\,,$$

$$k_{4,11} = \frac{3EI}{14} - \frac{151GA\kappa L^2}{4620}\,, \qquad k_{4,12} = \frac{-EIL}{84} + \frac{181GA\kappa L^3}{55440}\,,$$

$$k_{5,5} = \frac{8EIL}{35} + \frac{52GA\kappa L^3}{3465}\,, \qquad k_{5,6} = \frac{EIL^2}{60} + \frac{23GA\kappa L^4}{18480}\,,$$

$$k_{5,7} = -k_{2,10}\,, \qquad k_{5,8} = -k_{2,11}\,, \qquad k_{5,9} = -k_{2,12}\,, \qquad k_{5,10} = -k_{4,11}\,,$$

$$k_{5,11} = \frac{-(EIL)}{70} - \frac{19GA\kappa L^3}{1980}\,, \qquad k_{5,12} = \frac{EIL^2}{210} + \frac{13GA\kappa L^4}{13860}\,,$$

$$k_{6,6} = \frac{EIL^3}{630} + \frac{GA\kappa L^5}{9240}\,, \qquad k_{6,7} = -k_{3,10}\,, \qquad k_{6,8} = -k_{3,11}\,,$$

$$k_{6,9} = -k_{3,12}\,, \qquad k_{6,10} = k_{4,12}\,, \qquad k_{6,11} = -k_{5,12}\,,$$

$$k_{6,12} = \frac{EIL^3}{1260} + \frac{GA\kappa L^5}{11088}\,, \qquad k_{7,7} = k_{1,1}\,,$$

$$k_{7,8} = -k_{1,2}\,, \qquad k_{7,9} = k_{1,3}\,, \qquad k_{7,10} = -k_{1,4}\,, \qquad k_{7,11} = k_{1,5}\,,$$

$$k_{7,12} = -k_{1,6}\,, \qquad k_{8,8} = k_{2,2}\,, \qquad k_{8,9} = -k_{2,3}\,, \qquad k_{8,10} = k_{2,4}\,,$$

$$k_{8,11} = k_{2,5}\,, \qquad k_{8,12} = k_{2,6}\,, \qquad k_{9,9} = k_{3,3}\,, \qquad k_{9,10} = -k_{3,4}\,,$$

$k_{9,11} = k_{3,5}$, $k_{9,12} = k_{3,6}$, $k_{10,10} = k_{4,4}$, $k_{10,11} = -k_{4,5}$,
$k_{10,12} = k_{4,6}$, $k_{11,11} = k_{5,5}$, $k_{11,12} = -k_{5,6}$, $k_{12,12} = k_{6,6}$.

Finally, the soil stiffness matrix for the quintic element is given by:

$$k_{1,1} = \frac{(140k_{wl} + 41k_{wr})L}{462}, \qquad k_{1,2} = \frac{(644k_{wl} + 289k_{wr})L^2}{13860},$$

$$k_{1,3} = \frac{(182k_{wl} + 99k_{wr})L^3}{55440}, \qquad k_{1,4} = k_{1,5} = k_{1,6} = 0,$$

$$k_{1,7} = \frac{25(k_{wl} + k_{wr})L}{462}, \qquad k_{1,8} = \frac{-((239k_{wl} + 214k_{wr})L^2)}{13860},$$

$$k_{1,9} = \frac{(99k_{wl} + 82k_{wr})L^3}{55440}, \qquad k_{1,10} = k_{1,11} = k_{1,12} = 0,$$

$$k_{2,2} = \frac{(133k_{wl} + 75k_{wr})L^3}{13860}, \qquad k_{2,3} = \frac{(14k_{wl} + 9k_{wr})L^4}{18480},$$

$$k_{2,4} = k_{2,5} = k_{2,6} = 0, \qquad k_{2,7} = \frac{(214k_{wl} + 239k_{wr})L^2}{13860},$$

$$k_{2,8} = \frac{-19(k_{wl} + k_{wr})L^3}{3960}, \qquad k_{2,9} = \frac{(27k_{wl} + 25k_{wr})L^4}{55440},$$

$$k_{2,10} = k_{2,11} = k_{2,12} = 0, \qquad k_{3,3} = \frac{(7k_{wl} + 5k_{wr})L^5}{110880},$$

$$k_{3,4} = k_{3,5} = k_{3,6} = 0, \qquad k_{3,7} = \frac{(82k_{wl} + 99k_{wr})L^3}{55440},$$

$$k_{3,8} = \frac{-((25k_{wl} + 27k_{wr})L^4)}{55440}, \qquad k_{3,9} = \frac{(k_{wl} + k_{wr})L^5}{22176},$$

$$k_{3,10} = k_{3,11} = k_{3,12} = 0, \qquad k_{4,4} = \frac{(140k_{pl} + 41k_{pr})L}{462},$$

$$k_{4,5} = \frac{(644k_{pl} + 289k_{pr})L^2}{13860}, \qquad k_{4,6} = \frac{(182k_{pl} + 99k_{pr})L^3}{55440},$$

$$k_{4,7} = k_{4,8} = k_{4,9} = 0, \qquad k_{4,10} = \frac{25(k_{pl} + k_{pr})L}{462},$$

$$k_{4,11} = \frac{-((239k_{pl} + 214k_{pr})L^2)}{13860}, \qquad k_{4,12} = \frac{(99k_{pl} + 82k_{pr})L^3}{55440},$$

$$k_{5,5} = \frac{(133k_{pl} + 75k_{pr})L^3}{13860}, \qquad k_{5,6} = \frac{(14k_{pl} + 9k_{pr})L^4}{18480},$$

$$k_{5,7} = k_{5,8} = k_{5,9} = 0, \qquad k_{5,10} = \frac{(214k_{pl} + 239k_{pr})L^2}{13860},$$

$$k_{5,11} = \frac{-19(k_{pl}+k_{pr})L^3}{3960}, \qquad k_{5,12} = \frac{(27k_{pl}+25k_{pr})L^4}{55440},$$

$$k_{6,6} = \frac{(7k_{pl}+5k_{pr})L^5}{110880}, \qquad k_{6,7} = k_{6,8} = 0, \qquad k_{6,9} = 0,$$

$$k_{6,10} = \frac{(82k_{pl}+99k_{pr})L^3}{55440}, \qquad k_{6,11} = \frac{-((25k_{pl}+27k_{pr})L^4)}{55440},$$

$$k_{6,12} = \frac{(k_{pl}+k_{pr})L^5}{22176}, \qquad k_{7,7} = \frac{(41k_{wl}+140k_{wr})L}{462},$$

$$k_{7,8} = \frac{-((289k_{wl}+644k_{wr})L^2)}{13860}, \qquad k_{7,9} = \frac{(99k_{wl}+182k_{wr})L^3}{55440},$$

$$k_{7,10} = 0, \qquad k_{7,11} = k_{7,12} = 0, \qquad k_{8,8} = \frac{(75k_{wl}+133k_{wr})L^3}{13860},$$

$$k_{8,9} = \frac{-((9k_{wl}+14k_{wr})L^4)}{18480}, \qquad k_{8,10} = k_{8,11} = k_{8,12} = 0,$$

$$k_{9,9} = \frac{(5k_{wl}+7k_{wr})L^5}{110880}, \qquad k_{9,10} = k_{9,11} = k_{9,12} = 0,$$

$$k_{10,10} = \frac{(41k_{pl}+140k_{pr})L}{462}, \qquad k_{10,11} = \frac{-((289k_{pl}+644k_{pr})L^2)}{13860},$$

$$k_{10,12} = \frac{(99k_{pl}+182k_{pr})L^3}{55440}, \qquad k_{11,11} = \frac{(75k_{pl}+133k_{pr})L^3}{13860},$$

$$k_{11,12} = \frac{-((9k_{pl}+14k_{pr})L^4)}{18480}, \qquad k_{12,12} = \frac{(5k_{pl}+7k_{pr})L^5}{110880}.$$

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