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STATIONARY AND OSCILLATORY FLOW OF THE REINER-RIVLIN FLUID BETWEEN TWO EXTERNAL ROLLERS

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In a certain class of thin-layer viscoelastic flows, the extensional parts of deformation are much greater than the shearing ones. For such flows we may apply the model of the Reiner–Rivlin fluid [1]. This concept is applied to the stationary and oscillatory flow between two rotating external cylinders. The approximate solutions are presented and possible effects of the extensional viscosity function on the loads and friction forces are discussed.

1. Introduction

The problem of viscoelastic flows between two rotating rollers results from many practical applications to various lubricating systems, e.g. milling, rolling, calendering, etc. The theoretical analysis has been presented by many authors, e.g. compare the book by A. Cameron [2], but mainly for Newtonian or simple inelastic fluids. The number of publications in which the viscoelastic model of fluid is used is relatively small. In the present paper the problem of flow between two rotating cylinders is described by the constitutive equations of the Reiner - Rivlin fluid [1]. The approximate solutions of nonlinear equations of motion are obtained for slightly non-Newtonian fluids by means of the perturbation method for weak variability of the extensional viscosity function (cf. [1, 3]).

In this paper, being a direct development of our previous analysis (cf. [4, 5]), we consider the concept of steady and oscillatory flow between two cylinders. In the case of harmonic torsional vibrations superimposed on the steady motion of rollers we assume that the amplitudes and frequencies of vibrations are small, thus enabling rejection of inertia terms in the equation of motion.

2. Basic equations for flow of the Reiner–Rivlin fluid

Consider the plane steady flows in which the Cartesian velocity components can be expressed in the following form:

$$u^* = qx + u(x,y), \qquad v^* = -qy + v(x,y), \tag{2.1}$$

where q is some constant extension gradient, u, v denote the additional velocity components along the axis x and y, respectively.

We assume that the flow considered is realized in a thin layer characterised by the following small parameter

$$\varepsilon = \frac{h_0}{L} \ll 1, \tag{2.2}$$

where L, h_0 denote some characteristic lengths (in the x and y direction, respectively).

We apply the constitutive equations of the Reiner–Rivlin fluid in the form perturbed by an additional velocity gradient (cf. equations in [3]). In the case of plane flows these equations can be written as

$$T^* = -p\mathbf{1} + \beta\mathbf{A}_1 + \beta\mathbf{A}_1' + \frac{d\beta}{dq} q'\mathbf{A}_1\,, \tag{2.3}$$

where $\mathbf{A}_1$ is the Rivlin–Ericksen kinematic tensor, β – the material function, p – pressure and the primes denote increments of the corresponding quantities. In particular, we have

$$q' = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{1}{4q}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \frac{1}{8q}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\right)^2. \tag{2.4}$$

The above equations may be valid for the flow between two rollers under the assumptions that the extensional velocity gradients are more meaningful as compared with the shearing gradients (for small vorticity components or relatively high Deborah numbers, cf. [6]).

The procedure similar to that developed in [3] leads to the following, simplified equations of dynamic equilibrium:

$$\frac{dp^*}{dx} = \frac{1}{2}\frac{d\beta}{dq}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 + \beta\frac{\partial^2 u}{\partial y^2}\,, \qquad \frac{\partial p^*}{\partial y} = 0, \tag{2.5}$$

or, after eliminating the modified pressure p^*, we have alternatively

$$\frac{\partial}{\partial y}\left[\frac{1}{2}\frac{d\beta}{dq}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 + \beta\frac{\partial^2 u}{\partial y^2}\right] = 0. \tag{2.6}$$

The above equation is a third-order nonlinear partial differential equation, the exact solution of which is unknown and therefore we seek an approximate solution.

3. Flow between rollers. The steady case

3.1. Geometry

The geometry of rotating rollers is shown in Fig. 1. The rollers of the same radius R rotate with the constant tangential velocity V and the smallest distance between them is $2h_0$. According to A. Cameron [2], we have an approximate parabolic dependence describing the distance between cylinders in the form:

$$h = h_0\left(1 + \frac{x^2}{2}\right), \qquad L = \sqrt{2h_0 R}, \tag{3.1}$$

valid for small ratios x/L.

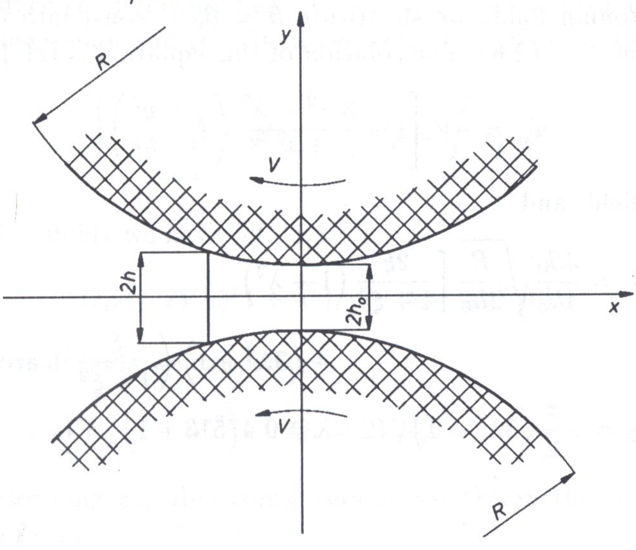

Fig. 1. Schematic diagram of the rollers geometry.

3.2. Boundary conditions

The kinematic boundary conditions result from the condition of fluid adherence to the surfaces of rollers. This leads to

$$u^* = -V \qquad \text{for} \quad y = \pm h. \tag{3.2}$$

At the exit cross-section, where the fluid leaves the rollers, a uniform velocity distribution satisfies the condition

$$u^* = -V = \text{const} \qquad \text{for} \quad x = -x_e\,. \tag{3.3}$$

Introducing dimensionless coordinates $\xi = x/l$ and $\lambda = x_e/l$, we obtain the following expressions for h and h_e:

$$h = h_0\left(1+\xi^2\right), \qquad h_e = h_0\left(1+\lambda^2\right). \tag{3.4}$$

We asume moreover that the modified pressure p^* satisfies the Reynolds-type boundary conditions (cf. [2])

$$\begin{aligned} &p^* = 0, \qquad \frac{dp^*}{dx} = 0, \qquad &&\text{for} \quad x = -x_e \\ &p^* = 0, && \text{for} \quad x \to \infty. \end{aligned} \tag{3.6}$$

3.3. The case of Newtonian fluid

For Newtonian fluids we substitute $\beta = \beta_0 = \text{const}$ into the nonlinear equation of motion (2.6). The solution of this equation is (cf. [2]):

$$u^*_N = -V\left[1 - \frac{3}{2}\frac{\xi^2-\lambda^2}{1+\xi^2}\left(1-\frac{y^2}{h^2}\right)\right], \tag{3.6}$$

for velocity field, and

$$\begin{aligned} p^*_N = \frac{3\beta_0}{4h_0}\sqrt{\frac{R}{2h_0}}&\left[\frac{2\xi}{1+\xi^2}\left(1+\lambda^2\right)\right. \\ &\left.+\left(3\lambda^2-1\right)\left(\frac{\xi}{1+\xi^2}+\operatorname{arc\,tg}\xi + C_2\right)\right], \\ C_2 = -\frac{\pi}{2}\left(3\lambda^2-1\right), &\qquad \lambda = 0.47513 \end{aligned} \tag{3.7}$$

for pressure.

3.4. The case of flow of the Reiner–Rivlin fluid

We shall seek an approximate distribution of u^* in the form:

$$u^* = u^*_N + ku_1, \tag{3.8}$$

where

$$u^*_N = qx + u_N - qx, \qquad k = q\frac{1}{\beta}\frac{d\beta}{dq}. \tag{3.9}$$

The form of Eq. (3.9) distinguishes the extension term qx. We assume moreover that $k \ll 1$. Substituting Eq. (3.9) into Eq. (3.8) we obtain

$$u^* = qx + u(x,y), \qquad u(x,y) = u_N - qx + k\,u_1, \tag{3.10}$$

and the equation of motion in the form:

$$\frac{\partial}{\partial y}\left[\frac{1}{2}\frac{1}{q}k\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2}\right] = 0. \tag{3.11}$$

Assuming that the orders of derivatives $\partial u_N/\partial y$, $\partial u_1/\partial y$ are identical, and taking into account that $\frac{\partial}{\partial y}(qx) = 0$, $\frac{\partial^3 u_n}{\partial y^3} = 0$, we arrive at the following equation of motion (we have neglected the terms of orders $0(k)$ and $0(k^2)$):

$$\frac{\partial}{\partial y}\left[\frac{1}{2}\frac{1}{q}k\frac{\partial}{\partial x}\left(\frac{\partial u_N}{\partial y}\right)^2 + k\frac{\partial^2 u_1}{\partial y^2}\right] = 0. \tag{3.12}$$

We introduce the following new quantities: the volumetric rates Q_N and Q^* for the Newtonian and real fluid, respectively, according to definitions:

$$Q_N = \int_{-h}^{+h} u_N\,dy, \qquad Q_N^* = \int_{-h}^{+h} u^*\,dy. \tag{3.13}$$

For Newtonian fluids we have (cf. [1]):

$$Q_N = 2Vh_0(1+\lambda^2), \qquad \lambda = 0.47513\,. \tag{3.14}$$

Similarly for real fluids we can write

$$Q^* = 2Vh_0\left(1+\lambda^{*2}\right), \quad Q_r = 2Vh_0\left(\lambda^{*^2}-\lambda^2\right), \quad Q_r = Q^* - Q_N. \tag{3.15}$$

After performing suitable computations we obtain the solution of Eq. (3.12) in the form:

$$u^* = qx + u_N + k\left[-\frac{1}{k}qx + BF_1y^4 - \left(\frac{3}{4}\frac{1}{kh^3}Q_r + \frac{6}{5}Bh^2F_1\right)y^2 + \frac{3}{4}\frac{1}{kh}Q_r + \frac{1}{5}Bh^4F_1\right], \tag{3.16}$$

where

$$B = \frac{3}{2}\frac{1}{q}\frac{V^2}{Lh_0^4}, \qquad F_1(\xi) = \xi\frac{2\xi^4 - \xi^2\left(1+5\lambda^2\right) + \left(\lambda^2+3\lambda^4\right)}{\left(1+\xi^2\right)^7}. \tag{3.17}$$

The kinematic boundary conditions

$$u_1 = 0 \quad \text{for} \quad y = \pm h, \tag{3.18}$$

resulting from Eq. (3.2) were applied in the relation (3.16). It is noteworthy that in Eq. (3.16) Q_r is an unknown function to be determined from the pressure boundary condition.

In order to find the pressure distribution, we use the first relation of the two equations of motion (2.5). Substituting into them the $u(x,y)$ field given by Eq. (3.10) and retaining, similarly as in Eq. (3.12), only the terms of higher order of magnitude, we obtain the relation

$$\frac{dp^*}{dx} = \frac{1}{2}\frac{d\beta}{dq}\frac{\partial}{\partial x}\left(\frac{\partial u_N}{\partial y}\right)^2 + \beta\left(\frac{\partial^2 u_N}{\partial y^2} + k\frac{\partial^2 u_N}{\partial y^2}\right). \tag{3.19}$$

Introducing into the above equation the known distributions of velocity $u(x,y)$ (cf. Eqs. (3.16), (3.17)) we have (putting $q = V/L$):

$$\frac{dp^*}{dx} = -\frac{3\beta V}{h_0^2}\left[\frac{\lambda^{*2}+\xi^2-2\lambda^2}{(1+\xi^2)^3} + 1.2k\xi\frac{2\xi^4-\xi^2\left(1+5\lambda^2\right)+\lambda^2+3\lambda^4}{(1+\xi^2)^5}\right]. \tag{3.20}$$

After integration we arrive at the following solutions:

$$p^* = \frac{L\beta V}{h_0^2}\left\{\frac{3\xi\left(1+\lambda^2\right)}{4\left(1+\xi^2\right)^2} - \frac{3}{8}\left(1+3\lambda^2\right)\left[\frac{\xi}{(1+\xi^2)} + \operatorname{arc\,tg}\xi - \frac{\pi}{2}\right] + k\left[1.8\frac{\xi^4}{(1+\xi^2)^4} + 0.07726\frac{1}{(1+\xi^2)^3} + 0.228339\frac{1}{(1+\xi^2)^4}\right]\right\}, \tag{3.21}$$

$$\lambda^* = \pm\lambda \qquad (Q_r = 0), \tag{3.22}$$

satisfying the Reynolds-type boundary conditions (3.5).

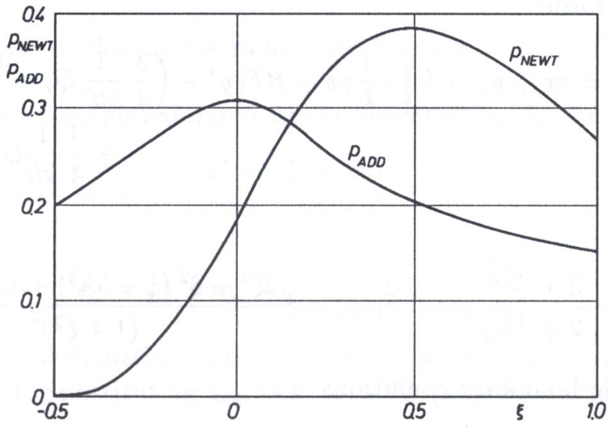

FIG. 2. Pressure distribution in the nip region between rollers.

This relation can be written as a sum of Newtonian and non-Newtonian terms in the form

$$p^* = \frac{L\beta V}{h_0^2}\left(p_{\mathrm{NEWT}} + k p_{\mathrm{ADD}}\right), \tag{3.23}$$

where p_{NEWT}, p_{ADD} obviously result from Eq. (3.21). The term $(L\beta V/h_0^2)\cdot p_{\mathrm{NEWT}}$ after formal transformation is identical with the well-known Newtonian distribution (cf. [2]). The above quantities are shown in Fig. 2.

3.5. Load and friction forces

Integration of the thrust from exit ($\xi = -\lambda^*$) to infinity

$$F^* = \int_{-\lambda^*}^{\infty} p^*(\xi)d\xi \tag{3.24}$$

leads to the total force (load capacity) in the form:

$$F^* = \frac{L\beta V}{h_0^2}\left(\int_{-\lambda}^{\infty} p_{\mathrm{NEWT}}\, d\xi + k\int_{-\lambda^*}^{\infty} p_{\mathrm{ADD}}\, d\xi\right), \tag{3.25}$$

or

$$F^* = \frac{L\beta V}{h_0^2}\left(F_{\mathrm{NEWT}} + kF_{\mathrm{ADD}}\right), \tag{3.26}$$

where the distributions of pressure p_{NEWT}, p_{ADD} are described by Eqs. (3.21), (3.23) and F_{NEWT}, F_{ADD} obviously result from Eq. (3.26).

We define the dimensionless parameter K_L being the measure of relative increase of the load capacity caused by the derivative $d\beta/dq$ as follows:

$$K_L = \frac{F_{\mathrm{NEWT}} + kF_{\mathrm{ADD}}}{F_{\mathrm{NEWT}}} = 1 + k\frac{F_{\mathrm{ADD}}}{F_{\mathrm{NEWT}}}. \tag{3.27}$$

After integration of the expression (3.25) we can calculate the parameter K_L. The diagram illustrating the variability of K_L as a function of k is shown in Fig. 3.

In a similar way we introduce the definition of total friction force in the form:

$$T^* = \int_{-\lambda}^{\infty} T_{12}^*(\xi)\Big|_{y=\pm h} d\xi. \tag{3.28}$$

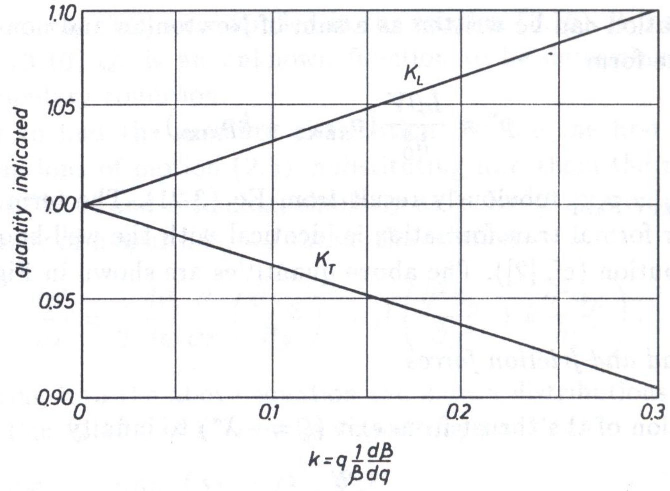

FIG. 3. Relative variability of the load capacity and the friction force.

According to [3] the shear stress can be expressed as

$$(3.29)\qquad T^*_{12} = \beta\frac{\partial u}{\partial y}.$$

Substituting into Eq. (3.28) the relationships (3.9) and (3.29), the total friction force T^* can be written as a sum of two integrals

$$(3.30)\qquad T^* = \int\limits_{-\lambda}^{\infty} \beta\frac{\partial u_N}{\partial y}\bigg|_{y=\pm h} d\xi + \int\limits_{-\lambda^*}^{\infty} \beta k\frac{\partial u_1}{\partial y}\bigg|_{y\pm h} d\xi,$$

where the velocity distribution u_N is given by Eq. (3.6) $[u^*_N = u_N]$ and the velocity field u_1 results from Eqs. (3.10), (3.16), (3.17).

After suitable calculations we arrive at

$$(3.31)\qquad T^* = -3\beta\frac{V}{h_0}\left(T_{\text{NEWT}} + kT_{\text{ADD}}\right),$$

where

$$(3.32)\qquad T_{\text{NEWT}} = \int\limits_{-\lambda}^{\infty} \frac{\xi^2-\lambda^2}{(1+\xi^2)^2}d\xi,$$

$$(3.33)\qquad T_{\text{ADD}} = -\frac{4}{5}\int\limits_{-\lambda^*}^{\infty} \xi\frac{2\xi^4-\xi^2\left(1+5\lambda^{*^2}\right)+\left(\lambda^{*^2}+3\lambda^{*^2}\right)}{(1+\xi^2)^4}d\xi.$$

The dimensionless parameter K_T, describing the variability of friction force caused by the extensional viscosity derivative with respect to the extension gradient, is expressed in the form

$$K_T = \frac{T_{\text{NEWT}} + kT_{\text{ADD}}}{T_{\text{NEWT}}} = 1 + k\frac{T_{\text{ADD}}}{T_{\text{NEWT}}} . \tag{3.34}$$

The variability of K_T as a function of the coefficient k is shown in Fig. 3.

4. Flow between rollers. Dynamic case

4.1. *Oscillatory boundary conditions*

We assume that the additional small-amplitude harmonic vibrations with the frequency ω are superimposed on a steady motion of cylinders. The vibrations can result from the oscillatory machine motion. We assume, moreover, that the fluid fully adhers to the surfaces of rollers. It leads to conditions:

$$u^*_\omega = -V + \epsilon(-V)\exp i\omega t \qquad \text{for} \quad y = \pm h, \tag{4.1}$$

where V denotes the tangential velocity on the surface of cylinders in the case of steady motion, subscript ω indicates the vibrating motion, ϵ is a small parameter characterizing the amplitude of vibrations.

At the cross-section, where the fluid leaves the rollers, a uniform velocity distribution satisfies the condition:

$$u^*_\omega = -V + \epsilon(-V)\exp i\omega t \qquad \text{for} \quad x = -x_e . \tag{4.2}$$

For modified pressure p^*_ω we superimpose the Reynolds-type boundary conditions, namely:

$$\begin{aligned} &p^*_\omega = 0, \qquad \frac{dp^*_\omega}{dx} = 0 \qquad &&\text{for} \quad x = -x_e, \\ &p^*_\omega = 0 &&\text{for} \quad x \to \infty. \end{aligned} \tag{4.3}$$

4.2. *The case of Newtonian fluid*

We consider the flow of Newtonian fluid between rotating cylinders in the case of harmonic vibrations. Similarly to the previous section, the same kinematic boundary conditions are satisfied:

$$u^\omega_N = -V + \epsilon(-V)\exp i\omega t \qquad \text{for} \quad y = \pm h, \tag{4.4}$$

where index N indicates Newtonian quantities.

In the Newtonian case (β = const) a solution can be obtained from Eq. (2.10). After suitable computations we arrive at following relation (cf. [7]):

$$u_N^{\omega} = u_N + \epsilon u_N \exp i\omega t, \tag{4.5}$$

where the velocity distribution u_N is given by Eq. (3.6).

4.3. The case of flow of the Reiner-Rivlin fluid

In an oscillatory motion of rollers we seek the velocity field u_ω^* in the following form:

$$u_\omega^* = qx + u_N^{\omega} - qx + k u_1^{\omega} \tag{4.6}$$

or, after introducing the obvious notation, in the form:

$$u_\omega^* = qx + u_\omega(x, y). \tag{4.7}$$

The unknown additional velocity u_1^ω consists of two parts: the first u_1 describing a steady motion of cylinders and the second $\overline{u}_1 \exp i\omega t$ depending on additional vibrations. Applying these assumptions we have

$$u_1^\omega = u_1 + \epsilon \overline{u}_1 \exp i\omega t, \tag{4.8}$$

where $\overline{u}_1$ is the sought amplitude of vibrations. Substituting Eq. (4.8) into Eq. (4.6) we arrive at

$$u_\omega^* = qx + u_N - qx + \epsilon u_N \exp i\omega t + k\,(u_1 + \epsilon\overline{u}_1 \exp i\omega t)\,. \tag{4.9}$$

Comparing Eqs. (4.7) and (4.9), we obtain

$$u_\omega(x, y) = u_N - qx + \epsilon u_N \exp i\omega t + k\,(u_1 + \overline{u}_1 \exp i\omega t)\,. \tag{4.10}$$

Looking for the unknown velocity distribution $\overline{u}_1$ we substitute Eq. (4.10) into Eq. (2.6) (instead of $u(x, y)$ we substitute $u_\omega(x, y)$). In this way, we obtain the following equation of motion:

$$\begin{aligned}\Big\{\frac{1}{2}\frac{k}{q}\Big[u^2_{N,y} &+ \epsilon^2 u^2_{N,y} \exp 2i\omega t + k^2 u^2_{1,y} + k^2\epsilon^2 \overline{u}_{1,y} \exp 2i\omega t \\ &+ 2\epsilon u^2_{N,y} \exp i\omega t + 2k u_{N,y} u_{1,y} + 2k\epsilon u_{N,y} \overline{u}_{1,y} \exp i\omega t \\ &+ 2k\epsilon u_{N,y} u_{1,y} \exp i\omega t + 2k\epsilon^2 u_{N,y} \overline{u}_{1,y} \exp 2i\omega t \\ &+ 2k^2 \epsilon u_{1,y} \overline{u}_{1,y} \exp i\omega t\Big]_{,x} + u_{N,yy} + \epsilon u_{N,yy} \exp i\omega t \\ &|+ k u_{1,yy} + k\epsilon \overline{u}_{1,yy} \exp i\omega t]_{,y} = 0,\end{aligned} \tag{4.11}$$

where partial differentiation is performed with respect to the variables appearing after the comma.

Simplifying the above equation we shall take into account the following relations and assumptions:

1) the partial derivatives of u_N, u_1, $\overline{u}_1$ are of the same orders of magnitude,

2) $\epsilon \ll 1$, $k \ll 1$, $0(\epsilon) = 0(k)$, $0(k\epsilon) = 0(\epsilon^2) = 0(k^2)$,

3) $u_{N,yyyy} = 0$ (cf. (3.6)),

4) $|\exp i\omega t| \le 1$, $|\exp 2i\omega t| \le 1$.

Omitting in Eq. (4.11) the terms of order of magnitude smaller than $0(\epsilon)$ and subtracting from this equation the statical relation

$$\left[\frac{1}{2}\frac{k}{q}\left(u^2_{N,y} + k^2 u_{1,y} + 2k u_{N,y} u_{1,y}\right)_{,x} + k u_{1,yy}\right]_{,y} = 0, \tag{4.12}$$

we arrive at the following expression:

$$\left[\frac{1}{2}\frac{k}{q} 2\epsilon (u_{N,y})^2_{,x} \exp i\omega t + k\epsilon \overline{u}_{1,yy} \exp i\omega t\right]_{,y} = 0 \tag{4.13}$$

or

$$\left[\frac{1}{q}(u_{N,y})^2_{,x} + \overline{u}_{1,yy}\right]_{,y} = 0. \tag{4.14}$$

The above equation is solved with the boundary conditions (4.1). Using these conditions in Eq. (4.9) we obtain

$$\Big| -V + \epsilon(-V) \exp i\omega t = \left[qx + u_N - qx + \epsilon u_N \exp i\omega t + k\left(u_1 + \epsilon \overline{u}_1 \exp i\omega t\right)\right]\Big|_{y=\pm h}. \tag{4.15}$$

Therefore the kinematic boundary conditions, in view of $u^*_N = -V$, $u_1 = 0$ for $y = \pm h$ (cf. (3.11)), are simplified to the relations

$$\overline{u}_1 = 0 \qquad \text{for} \quad y = \pm h. \tag{4.16}$$

After performing the suitable calculations we come to the following form of the velocity distribution u^*_ω:

$$u^*_\omega = qx + u_N - qx + \epsilon u_N \exp i\omega t + k\left(u_1 + 2u_1 \exp i\omega t\right). \tag{4.17}$$

The velocity distribution presented in Eq. (4.17) satisfies not only the kinematic boundary conditions but also the first condition (4.3) for the pressure p^*_ω as well.

In place of the velocity field $u(x,y)$ in Eq. (2.5) we substitute the dynamic velocity distribution $u_\omega(x,y)$ described by relations (4.7) and (4.9). In such a way we obtain

$$(4.18)\qquad \frac{dp^*_\omega}{dx} = \frac{1}{2}\frac{d\beta}{dq}\Big(u^2_{N,y} + \epsilon^2 u^2_{N,y}\exp 2i\omega t + k^2 u_{1,y} + \epsilon^2 k^2 \overline{u}_{1,y}\exp 2i\omega t + 2\epsilon u^2_{N,y}\exp i\omega t + 2k u_{N,y} u_{1,y} + 2k\epsilon u_{N,y}\overline{u}_{1,y}\exp i\omega t + 2k\epsilon u_{N,y} u_{1,y}\exp i\omega t + 2k\epsilon^2 u_{N,y}\overline{u}_{1,y}\exp 2i\omega t + 2k^2\epsilon u_{1,y}\overline{u}_{1,y}\exp i\omega t\Big)_{,x} + \beta u^2_{N,y} + \beta\epsilon u_{N,yy}\exp i\omega t + \beta k u_{1,yy} + \beta\epsilon k\overline{u}_{1,yy}\exp i\omega t.$$

Bearing in mind the problem of steady motion of the cylinders we determine the analogous equation for the pressure p^*_{ST} in the following form:

$$(4.19)\qquad \frac{dp^*_{\mathrm{ST}}}{dx} = \frac{1}{2}\frac{d\beta}{dq}\left(u^2_{N,y} + k u_{1,y} + 2k u_{1,y} u_{N,y}\right)_{,x} + \beta u^2_{N,yy} + \beta k u^2_{1,yy}.$$

For further calculations we treat the dynamic modified pressure p^*_ω as a sum of two terms, namely: the statical part p^*_{ST} and the dynamical part p^*_{DYN}. This leads to the expression

$$(4.20)\qquad p^*_\omega = p^*_{\mathrm{ST}} + \epsilon p^*_{\mathrm{DYN}}\exp i\omega t.$$

Subtracting Eq. (4.19) from Eq. (4.18) and disregarding the terms of order $0(\epsilon^2)$, $0(\epsilon^3)$, $0(\epsilon^4)$ as compared to the terms of order $0(\epsilon)$, we obtain the expression for derivative dp^*_{DYN}/dx

$$(4.21)\qquad \frac{dp^*_{\mathrm{DYN}}}{dx} = \left[\frac{1}{q}k\beta\epsilon(u_{N,y})^2_x + \beta\epsilon u_{N,yy} + k\epsilon\beta\overline{u}_{1,yy}\right]\exp i\omega t.$$

After suitable calculations we arrive at the final solution

$$(4.22)\qquad p^*_\omega = \frac{L\beta V}{h_0^2}\left[p_{\mathrm{NEWT}} + k p_{\mathrm{ADD}}\right] + \epsilon\exp i\omega t\left[p_{\mathrm{NEWT}} + 2k p_{\mathrm{ADD}}\right].$$

The pressure distribution p^*_ω satisfies the Reynolds-type boudary condition.

Comparing the Eqs. (4.20) and (4.22) we find that

$$(4.23)\qquad p^*_{\mathrm{ST}} = \frac{L\beta V}{h_0^2}\left[p_{\mathrm{NEWT}} + k p_{\mathrm{ADD}}\right],$$

$$(4.24)\qquad p^*_{\mathrm{DYN}} = \frac{L\beta V}{h_0^2}\left[p_{\mathrm{NEWT}} + 2k p_{\mathrm{ADD}}\right].$$

We define the dimensionless parameter K being the measure of relative increase of the pressure caused by the additional harmonic vibrations of the rollers in the form:

$$K = \frac{p^*_{\mathrm{DYN}}}{p^*_{\mathrm{ST}}} = \frac{1 + 2k\dfrac{p_{\mathrm{ADD}}}{p_{\mathrm{DYN}}}}{1 + k\dfrac{p_{\mathrm{ADD}}}{p_{\mathrm{DYN}}}}. \tag{4.25}$$

The diagrams illustrating the variability of K as a function of the variable ξ for two values of the coefficient k are shown in Fig. 4.

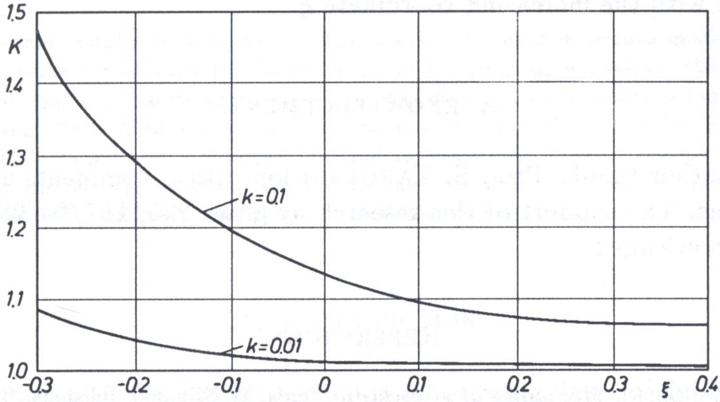

FIG. 4. Relative increase of pressure caused by harmonic vibrations.

5. Conclusions

1. The variability of extensional viscosity function $d\beta/dq$ essentialy influences the pressure and velocity distribution in the nip flow between the rotating cylinders.

2. The x-coordinate describing the point of uniform velocity distribution for non-Newtonian fluid is the same as in the Newtonian case, i.e. $\lambda^* = \lambda = 0.47513$ and does not depend on variability of the viscosity function β.

3. The additional term of the modified pressure p_{ADD} caused by the derivative $d\beta/dq$ effects increases the whole pressure p^*. The significant value of the additional p_{ADD} is reached in the neighbourhood of the smallest distance between rollers (cf. Fig. 2).

4. All the results depend on extensional viscosity function and its variability with increasing extension rates. The ratio of the extensional viscosity derivative with respect to the extension gradient to the extensional viscosity

itself, may be defined as the nonlinearity coefficient. In a steady case the load capacity of the rollers described by parameter K_L linearly increases with increasing nonlinearity coefficient, whereas the friction force characterized by parameter K_T linearly decreases (cf. Fig. 3).

5. The additional vibrations superimposed on a steady motion of cylinders not only change the extensional viscosity effect on the pressure and velocity distribution, but also contribute to further changes of the velocity field and to the enhancement of pressure in comparison with the case of steady motion of rollers. The pressure increment which is described by the parameter K is greater for higher values of the nonlinearity coefficient and decreases with the increasing coordinate ξ.

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