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NONLINEAR VISCOELASTIC ELLIPTICAL BAR UNDER COMBINED TORSION WITH TENSION

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In [1] DARWISH and CANNON considered the problem of twisting of a cylindrical bar of elliptic cross-section. The bar was assumed to consist of a viscoelastic material. The principal cubic theory of nonlinear viscoelasticity was used to obtain a nonlinear Volterra integral equation of the second kind for the angle of twist $\theta(t)$ as a function of time t. In this work we consider a cylindrical bar of elliptic cross-section which consists of a viscoelastic material. At first, we subject the bar to torsion around the axis of the bar as we did in the previous work [1]. After an interval of time, we subject the bar to a longitudinal axial tensile force, in addition to torsion, for an additional period of time. We derive a system of two nonlinear Volterra integral equations of the second kind for the angle of twist $\theta(t)$ and the relative extension function $\gamma(t)$. We prove the existence and uniqueness of the solution pair $(\theta(t), \gamma(t))$ and analyze a numerical procedure for the approximate solution of the pair $(\theta(t), \gamma(t))$. Results of a numerical study of $(\theta(t), \gamma(t))$ are presented for both the linear and nonlinear theory. The behaviour of the normal axial stress and the shearing stresses are also investigated for each pair $(\theta(t), \gamma(t))$, and the results for both the linear and nonlinear theory are presented.

1. INTRODUCTION

In this article we consider a long homogeneous isotropic bar with a constant elliptical cross-section. We assume that the bar consists of a nonlinear viscoelastic material. The bar is first subjected to a twisting moment at its ends for a certain time interval $0 \leq t \leq t_1$. Then for $t > t_1$, the bar is subjected to the twisting couple combined with an axial longitudinal tension force in the positive direction of the axis of the bar, which is taken to be the positive z-axis. We assume that the cross-section is constant along the bar and rotates uniformly with axial distance. We also assume that each cross-section undergoes the same deformation.

FINDLEY *et al.* [4, 5, 6] have carried out many experiments on some viscoelastic materials under combined tension and torsion. DARWISH [3] has considered the case of a circular bar of nonlinear viscoelastic material under an abrupt change in the state of stress. For elastic bars of elliptical cross-section, SHIELD [7] has considered the combination of tension and torsion.

The present article extends the work of DARWISH and CANNON [1] in which a nonlinear viscoelastic bar of elliptic cross-section is subjected to a twisting moment for a time interval $0 \leq t \leq t_1$. This produces an angle of twist $\theta(t)$ which is obtained as the solution of a nonlinear Volterra integral equation of the second kind. This is assumed as the first stage of the problem here. For $t > t_1$, we consider the combination of a twisting moment from the first stage with an axial longitudinal tensile force. The effect of the tension leads to the relative extension function $\gamma(t)$ which is a function of t only like $\theta(t)$. For $t > t_1$, we derive a system of two nonlinear Volterra integral equations of the second kind for the pair $(\theta(t), \gamma(t))$. We prove the existence and the uniqueness of the solution pair $(\theta(t), \gamma(t))$ and analyze a numerical procedure for the estimation of the pair. For a few values of torsion and tension, we calculate numerical approximations to the pair $(\theta(t), \gamma(t))$ for both the linear and nonlinear cases. We display the graphs of both cases for comparison. Using these results, we investigate numerically the behaviour of the normal axial stress and the shear stress. Finally, we conclude with a discussion of the numerical results and the comparison between the linear and nonlinear theory.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

We assume that the bar is subjected to no body forces and is free of forces acting on its lateral surface. We shall assume that the bar is of length l and that one end of the bar is fixed in the plane $z = 0$, with the axis of the bar extending along the positive z-axis from $z = 0$ to $z = l$. We shall assume that the bar has a uniform elliptic cross-section. We shall take the major axis of the elliptical cross-section to coincide with the x-axis from $x = -a$ to $x = a$, while the minor extends along the y-axis from $y = -b$ to $y = b$ (Fig.1). Hence, the boundary of any cross-section of the bar is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2.1}$$

In [1] DARWISH and CANNON derived the formulas for the shear stresses in an elliptical viscoelastic bar under torsion. So, we consider here the combination of torsion $M(t)$ with an axial longitudinal tension $T(t)$ applied to

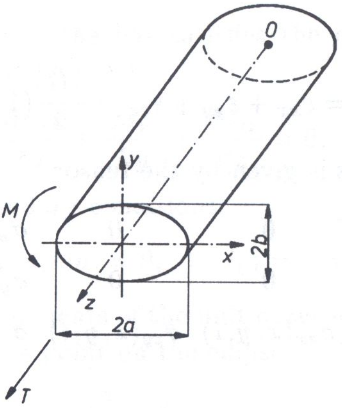

FIG. 1.

the end of the bar at $z = l$, and acting in the positive direction of the z-axis. The state of strain for this case is given by the tensor

$$(\varepsilon_{ij}(x,y,t)) = \begin{bmatrix} \varepsilon_{xx}(x,y,t) & 0 & \varepsilon_{xz}(x,y,t) \\ 0 & \varepsilon_{yy}(x,y,t) & \varepsilon_{yz}(x,y,t) \\ \varepsilon_{zx}(x,y,t) & \varepsilon_{zy}(x,y,t) & \varepsilon_{zz}(x,y,t) \end{bmatrix}, \tag{2.2}$$

where

$$\varepsilon_{zz}(x,y,t) = \frac{\gamma(t)}{2}, \tag{2.3}$$

and

$$\varepsilon_{xx}(x,y,t) = \varepsilon_{yy}(x,y,t) = -\frac{\nu}{2}\gamma(t), \tag{2.4}$$

and where

$$\varepsilon_{yz}(x,y,t) = \frac{\theta(t)}{2}\left(\frac{\partial\varphi(x,y)}{\partial y} + x\right), \tag{2.5}$$

and

$$\varepsilon_{zx}(x,y,t) = \frac{\theta(t)}{2}\left(\frac{\partial\varphi(x,y)}{\partial x} - y\right). \tag{2.6}$$

Here $\gamma(t)$ is the relative extension of the bar, ν is Poisson's ratio of the material, $\theta(t)$ is the angle of twist, and $\varphi(x,y)$ is the torque function (called also the warping function) and, according to SOKOLNIKOFF [8],

$$\varphi(x,y) = -\frac{a^2 - b^2}{a^2 + b^2}xy. \tag{2.7}$$

In addition,

$$\varepsilon_{kk} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\gamma(t)}{2}(1-2\nu). \tag{2.8}$$

Now, the state of stress is given by the tensor

$$(\sigma_{ij}) = \begin{bmatrix} 0 & 0 & \sigma_{xz}(x,y,t) \\ 0 & 0 & \sigma_{yz}(x,y,t) \\ \sigma_{zx}(x,y,t) & \sigma_{zy}(x,y,t) & \sigma_{zz}(x,y,t) \end{bmatrix}, \tag{2.9}$$

where

$$\sigma_{kk} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_{zz}\,. \tag{2.10}$$

The deviatoric strain tensor is given by

$$e_{ij}(x,y,t) = \varepsilon_{ij}(x,y,t) - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\,, \qquad i,j = x,y,z, \tag{2.11}$$

where δ_{ij} is the Kronecker delta function such that $\delta_{ij} = 0$, $i \neq j$, and $\delta_{ij} = 1$, $i = j$. Likewise, the deviatoric stress tensor is given by

$$S_{ij}(x,y,t) = \sigma_{ij}(x,y,t) - \frac{1}{3}\sigma_{kk}(x,y,t)\delta_{ij}\,, \qquad i,j = x,y,z, \tag{2.12}$$

and the second invariant of the strain deviator is given by

$$e^2(x,y,t) = \varepsilon_{ij}(x,y,t)\varepsilon_{ij}(x,y,t) - \frac{1}{3}\varepsilon_{kk}\varepsilon_{kk}\,, \tag{2.13}$$

where the summation convention holds for the repeated i and j subscripts, while the last term in (2.13) denotes minus one third of the square of the sum ε_{kk} defined in (2.8). Now, the principal cubic theory of nonlinear viscoelasticity [2] postulates a stress-strain constitutive relation given by

$$\begin{aligned} S_{ij}(x,y,t) = 2Ge_{ij}(x,y,t) &+ \int_0^t J(t-\tau)e_{ij}(x,y,\tau)\,d\tau \\ &+ \int_0^t K(t-\tau)e^2(x,y,\tau)e_{ij}(x,y,\tau)\,d\tau, \qquad i,j = x,y,z, \end{aligned} \tag{2.14}$$

where G is the instantaneous shear modulus of the material, and $J(t)$ and $K(t)$ are the respective kernel functions for the linear and nonlinear relaxation functions of the material; functions $J(t)$ and $K(t)$ will be given below.

The stress state (2.9) of the bar satisfies the equation of equilibrium $\sigma_{ij,j} = 0$ which in this case becomes

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0, \tag{2.15}$$

and is subject to the boundary conditions

$$\sigma_{ij}(x, y, t) n_j = 0, \qquad i, j = x, y, z, \tag{2.16}$$

where n_j denotes the components of the unit outer normal, and where (x, y) denote the coordinates of a point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The state of strain (2.2) in the bar satisfies the compatibility equations

$$\frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial \varepsilon_{yz}}{\partial x} = -\theta(t) \tag{2.17}$$

and

$$\frac{\partial \varepsilon_{xz}}{\partial x} - \frac{\partial \varepsilon_{yz}}{\partial y} = 0. \tag{2.18}$$

3. Derivation of the system of integral equations for $\theta(t)$ and $\gamma(t)$

For the nonvanishing components of deviatoric strain we obtain from (2.3)–(2.8) and (2.11) the following results:

$$\begin{aligned}
e_{yz}(x, y, t) &= \varepsilon_{yz}(x, y, t) = \frac{\theta(t)}{2}\left(\frac{\partial \varphi}{\partial y} + x\right), \\
e_{zx}(x, y, t) &= \varepsilon_{zx}(x, y, t) = \frac{\theta(t)}{2}\left(\frac{\partial \varphi}{\partial x} - y\right), \\
e_{xx}(x, y, t) &= \varepsilon_{xx}(x, y, t) - \frac{1}{3}\varepsilon_{kk} = -\frac{\nu}{2}\gamma(t) - \frac{1}{3}\frac{\gamma(t)}{2}(1 - 2\nu) \\
&= -\frac{1}{6}\gamma(t)(1 + \nu), \\
e_{yy}(x, y, t) &= \varepsilon_{yy}(x, y, t) - \frac{1}{3}\varepsilon_{kk} = -\frac{1}{6}\gamma(t)(1 + \nu), \\
e_{zz}(x, y, t) &= \varepsilon_{zz}(x, y, t) - \frac{1}{3}\varepsilon_{kk} = \frac{\gamma(t)}{2} - \frac{1}{3}\frac{\gamma(t)}{2}(1 - 2\nu) \\
&= \frac{\gamma(t)}{3}(1 + \nu),
\end{aligned} \tag{3.1}$$

and from (2.13) we obtain

$$(3.2)\qquad e^2(x,y,t)=\frac{1}{6}\left\{(1+\nu)^2\gamma^2(t)+3\theta^2(t)\left[\left(\frac{\partial\varphi}{\partial y}+x\right)^2+\left(\frac{\partial\varphi}{\partial x}-y\right)^2\right]\right\}.$$

Here $\varphi(x,y)$ is given by (2.7). Now, from Eqs. (2.9), (2.10) and (2.12) we see that

$$(3.3)\qquad \begin{aligned} S_{xz}(x,y,t) &= \sigma_{xz}(x,y,t),\\ S_{yz}(x,y,t) &= \sigma_{yz}(x,y,t),\\ S_{xx}(x,y,t) &= -\frac{1}{3}\sigma_{zz}(x,y,t),\\ S_{yy}(x,y,t) &= -\frac{1}{3}\sigma_{zz}(x,y,t),\\ S_{zz}(x,y,t) &= \frac{2}{3}\sigma_{zz}(x,y,t). \end{aligned}$$

Substituting $(3.1)_5$, (3.2), $(3.3)_5$ into the stress-strain constitutive relation (2.14), we obtain

$$(3.4)\qquad \begin{aligned} \sigma_{zz}(x,y,t) &= 2G(1+\nu)\frac{\gamma(t)}{2}+\frac{(1+\nu)}{2}\int_0^t J(t-\tau)\gamma(t)\,d\tau\\ &\quad+\frac{(1+\nu)}{2}\int_0^t K(t-\tau)e^2(x,y,t)\gamma(\tau)\,d\tau\\ &= E\frac{\gamma(t)}{2}+\frac{(1+\nu)}{2}\int_0^t J(t-\tau)\gamma(\tau)\,d\tau\\ &\quad+\frac{(1+\nu)}{12}\int_0^t K(t-\tau)\left\{(1+\nu)^2\gamma^2(\tau)\right.\\ &\quad\left.+3\theta^2(\tau)\left[\left(\frac{\partial\varphi}{\partial y}+x\right)^2+\left(\frac{\partial\varphi}{\partial x}-y\right)^2\right]\right\}\gamma(\tau)\,d\tau, \end{aligned}$$

where

$$(3.5)\qquad E=2G(1+\nu)$$

is the instantaneous value of Young's modulus. Next, we substitute $(3.1)_5$, (3.2) and $(3.3)_2$ into the stress-strain constitutive relation (2.14) and we obtain

$$\sigma_{yz}(x,y,t) = \left(\frac{\partial\varphi}{\partial y} + x\right)\left[G\theta(t) + \frac{1}{2}\int_0^t J(t-\tau)\theta(\tau)\,d\tau \right. \tag{3.6}$$

$$\left. + \frac{1}{12}\int_0^t K(t-\tau)\theta(\tau)\left\{(1+\nu)^2\gamma^2(\tau) + 3\theta^2(\tau)\left[\left(\frac{\partial\varphi}{\partial y} + x\right)^2 + \left(\frac{\partial\varphi}{\partial x} - y\right)^2\right]\right\}d\tau\right].$$

Likewise, using $(3.1)_2$, (3.2), $(3.3)_1$ and the stress-strain constitutive relation (2.14), we see that

$$\sigma_{xz}(x,y,t) = \left(\frac{\partial\varphi}{\partial x} - y\right)\left[G\theta(t) + \frac{1}{2}\int_0^t J(t-\tau)\theta(\tau)\,d\tau \right. \tag{3.7}$$

$$\left. + \frac{1}{12}\int_0^t K(t-\tau)\theta(\tau)\left\{(1+\nu)^2\gamma^2(\tau) + 3\theta^2(\tau)\left[\left(\frac{\partial\varphi}{\partial y} + x\right)^2 + \left(\frac{\partial\varphi}{\partial x} - y\right)^2\right]\right\}d\tau\right].$$

We note that we can obtain the stresses σ_{xz}, σ_{yz} and σ_{zz} for the linear theory by taking $K(t) = 0$ in Eqs. (3.4), (3.6) and (3.7).

From the equilibrium considerations we obtain

$$T(t) = \iint_R \sigma_{zz}(x,y,t)\,dx dy, \tag{3.8}$$

and

$$M(t) = \iint_R (x\sigma_{yz} - y\sigma_{zx})\,dx dy, \tag{3.9}$$

where R is the region bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Substituting the right-hand side of (3.4) into (3.8) we obtain, after some elementary calculations,

$$T(t) = \frac{1}{2}E\gamma(t)I_0 + \frac{(1+\nu)}{2}I_0\int_0^t J(t-\tau)\gamma(\tau)\,d\tau \tag{3.10}$$

$$+ \frac{(1+\nu)^3}{12}I_0\int_0^t K(t-\tau)\gamma^3(\tau)\,d\tau + \frac{(1+\nu)}{4}I_1\int_0^t K(t-\tau)\gamma(\tau)\theta^2(\tau)\,d\tau,$$

where

$$I_0 = \iint_R dx dy = \pi ab, \tag{3.11}$$

and

$$I_1 = \iint_R \left[\left(\frac{\partial \varphi}{\partial y} + x\right)^2 + \left(\frac{\partial \varphi}{\partial x} - y\right)^2\right] dx dy = \frac{\pi a^3 b^3}{(a^2 + b^2)}. \tag{3.12}$$

Substituting the right-hand sides of (3.6) and (3.7) into (3.9), we obtain, after elementary transformations,

$$M(t) = G\theta(t) I_2 + \frac{1}{2} I_2 \int_0^t J(t-\tau)\theta(\tau)\, d\tau + \frac{1}{4} I_3 \int_0^t K(t-\tau)\theta^3(\tau)\, d\tau + \frac{(1+\nu)^2}{12} I_2 \int_0^t K(t-\tau)\gamma^2(\tau)\theta(\tau)\, d\tau, \tag{3.13}$$

where

$$I_2 = \iint_R \left[\left(x\frac{\partial \varphi}{\partial y} - y\frac{\partial \varphi}{\partial x}\right) + (x^2 + y^2)\right] dx dy = \frac{\pi a^3 b^3}{(a^2 + b^2)}, \tag{3.14}$$

and

$$I_3 = \iint_R \left[\left(x\frac{\partial \varphi}{\partial x} - y\frac{\partial \varphi}{\partial x}\right) + (x^2 + y^2)\right]\left[\left(\frac{\partial \varphi}{\partial y} + x\right)^2 + \left(\frac{\partial \varphi}{\partial x} - y\right)^2\right] dx dy = \frac{4\pi a^5 b^5}{3(a^2 + b^2)^2}. \tag{3.15}$$

In [2], DARWISH obtained for polyurethane the following forms for the relaxation kernels:

$$J(t) = -Amt^{m-1} \tag{3.16}$$

and

$$K(t) = -Bmt^{m-1}, \tag{3.17}$$

where

$$m = 0.125, \qquad A = 0.217 \times 10^4, \tag{3.18}$$

and

$$B = 0.085 \times 10^8 \tag{3.19}$$

with units of kilogram-force per square centimeter-hour. Using the results (3.10) through (3.19), we are able to rewrite (3.10) and (3.13) in the form

$$\theta(t) = \alpha(t) + \int_0^t m(t-\tau)^{m-1}\left\{\beta\theta(\tau) + \delta\theta^3(\tau) + \varrho\theta(\tau)\gamma^2(\tau)\right\} d\tau, \tag{3.20}$$

and

$$\gamma(t) = \mu(t) + \int_0^t m(t-\tau)^{m-1}\left\{\zeta\gamma(\tau) + \eta\gamma^3(\tau) + \xi\gamma(\tau)\theta^2(\tau)\right\} d\tau, \tag{3.21}$$

where

$$\begin{aligned}
\alpha(t) &= M(t)/(GI_2),\\
\beta &= \frac{A}{2G},\\
\delta &= \frac{I_3 B}{4GI_1},\\
\varrho &= \frac{B(1+\nu)^2}{12G},\\
\mu(t) &= 2T(t)/(EI_0) = T(t)/(G(1+\nu)I_0),\\
\zeta &= \frac{A(1+\nu)}{E} = \frac{A}{2G},\\
\eta &= \frac{B(1+\nu)^3}{6E} = \frac{B(1+\nu)^2}{12G},\\
\xi &= \frac{BI_1}{4GI_0},
\end{aligned} \tag{3.22}$$

and

$$2G = 0.264 \times 10^5 \tag{3.23}$$

in the units of kilogram-force per square centimeter. We note that (3.20) and (3.21) is a nonlinear system of Volterra integral equations of the second kind.

REMARK. The value for ν for polyurethane is found below to be 0.389, which yields a value for E of 0.367×10^5.

4. THE SYSTEM OF INTEGRAL EQUATIONS

In this section we shall discuss the system of integral equations

$$\theta(t) = \alpha(t) + \int_0^t m(t-\tau)^{m-1}\left\{\beta\theta(\tau) + \delta\theta^3(\tau) + \varrho\theta(\tau)\gamma^2(\tau)\right\} d\tau,$$

and

$$\gamma(t) = \mu(t) + \int_0^t m(t-\tau)^{m-1} \left\{ \zeta\gamma(\tau) + \eta\gamma^3(\tau) + \xi\gamma(\tau)\theta^2(\tau) \right\} d\tau,$$

and the numerical approximation of its solution $(\theta(t), \gamma(t))$, where $m, \beta, \delta, \varrho, \zeta, \eta$ and ξ are known positive constants, and where $\alpha(t)$ and $\mu(t)$ are known functions of time t. It is well known that there exists an interval $0 \le t \le k$ for reasonable $\alpha(t)$ and $\mu(t)$ such that the system of integral equations possesses a unique solution pair $(\theta(t), \gamma(t))$ which depends continuously upon the data m, β, δ, ϱ, ζ, η, ξ, $\alpha(t)$ and $\mu(t)$. We shall sketch the essential part of the argument here, since we must use it in our discussion of the numerical method.

Consider the mapping

$$(4.1)\qquad \begin{aligned} (F\theta)(t) &= \alpha(t) + \int_0^t m(t-\tau)^{m-1} \left\{ \beta\theta(\tau) + \delta\theta^3(\tau) + \varrho\theta(\tau)\gamma^2(\tau) \right\} d\tau, \\ (F\gamma)(t) &= \mu(t) + \int_0^t m(t-\tau)^{m-1} \left\{ \zeta\gamma(\tau) + \eta\gamma^3(\tau) + \xi\gamma(\tau)\theta^2(\tau) \right\} d\tau, \end{aligned}$$

of the continuous functions $\theta(t)$ and $\gamma(t)$ defined in the interval $0 \le t \le k$, where k will be determined below. If $\alpha(t)$ and $\mu(t)$ are continuous, then there is a positive constant C such that

$$(4.2)\qquad |\alpha(t)| \le C,$$

and

$$(4.3)\qquad |\mu(t)| \le C,$$

for $0 \le t \le k$. Then, for the continuous functions $\theta(t)$ and $\gamma(t)$ on $0 \le t \le k$, which are bounded in absolute value by $2C$, we see that $(F\theta)(t)$ and $(F\gamma)(t)$ are equi-continuous for all such $\theta(t)$ and $\gamma(t)$ and that for $0 \le t \le k$

$$(4.4)\qquad |(F\theta)(t)| \le C + k^m \left\{ 2\beta C + 8(\delta + \varrho)C^3 \right\},$$

and

$$(4.5)\qquad |(F\gamma)(t)| \le C + k^m \left\{ 2\zeta C + 8(\eta + \xi)C^3 \right\}.$$

Selecting k so that

$$(4.6)\qquad k < \min \left\{ \left\{ 2\beta + 8(\delta + \varrho)C^2 \right\}^{-1/m}, \left\{ 2\zeta + 8(\eta + \xi)C^2 \right\}^{-1/m} \right\},$$

we find that for $0 \le t \le k$,

$$|(F\theta)(t)| \le 2C,$$

and

$$|(F\gamma)(t)| \le 2C.$$

Thus, as the set of continuous function pairs $(\theta(t), \gamma(t))$ on $0 \le t \le k$, satisfying $|\theta(t)| \le 2C$ and $|\gamma(t)| \le 2C$, is convex and closed under any uniform vector norm, and F is a continuous operator on that set whose image is precompact and contained in that set, from Schauder's Fixed Point Theorem [10] follows the existence of a fixed point $(\theta(t), \gamma(t)) = ((F\theta)(t), (F\gamma)(t))$, $0 \le t \le k$.

Let

$$\|(\theta(t), \gamma(t))_k\| = \max\{\|\theta\|_k, \|\gamma\|_k\},$$

where

$$\|\chi\|_k = \sup_{0 \le t \le k} |\chi(t)|.$$

Then, we see that

$$\begin{aligned} (4.7)\qquad & \|(F\theta_1, F\gamma_1) - (F\theta_2, F\gamma_2)\|_k \\ & \le k^m \max\Big\{\big\{\beta + 12(\delta + \varrho)C^2\big\}, \big\{\zeta + 12(\eta + \xi)C^2\big\}\Big\}\|(\theta_1, \gamma_1) - (\theta_2, \gamma_2)\|_k \end{aligned}$$

for $\theta_1, \gamma_1, \theta_2$, and γ_2 continuous for $0 \le t \le k$ and bounded in absolute value by $2C$. Thus, if

$$(4.8)\qquad k < \min\left\{\big\{\beta + 12(\delta + \varrho)C^2\big\}^{-1/m}, \big\{\zeta + 12(\eta + \xi)C^2\big\}^{-1/m}\right\},$$

then F is a contraction, and by a continuation argument the fixed point $(\theta, \gamma) = (F\theta, F\gamma)$ is unique over its interval of existence.

One point remains to be discussed in this section. For our application, $\mu(t) \equiv 0$, $0 \le t \le t_1 < k$. In this case, it follows that, for all t satisfying $0 \le t \le t_1 < k$,

$$(4.9)\qquad |\gamma(t)| \le t^m\big\{\zeta + 4C^2(\eta + \xi)\big\}|\gamma(t)|.$$

From (4.6) it follows that there exists a positive number κ, $0 < \kappa < 1$, such that for $0 \le t \le t_1 < k$,

$$(4.10)\qquad |\gamma(t)| \le \kappa|\gamma(t)|,$$

which implies that $\gamma(t) \equiv 0$ for $0 \le t \le t_1 < k$. We summarize this result in the following statement.

LEMMA 1. If $\mu(t) \equiv 0$, $0 \le t \le t_1 < k$, then $\gamma(t) \equiv 0$ for $0 \le t \le t_1 < k$.

Proof. See the discussion preceding the statement of the Lemma.

5. Numerical method

We turn now to a numerical method of approximating $\theta(t)$ and $\gamma(t)$. Let k be a given positive number that satisfies (4.6) and (4.8). Let N be a positive integer. Set $h = k/N$ and let $t_i = ih$, $i = 0, \ldots, N$, denote grid points. At t_i we see that

$$(5.1) \qquad \theta(t_i) = \alpha(t_i) + \int_0^{t_i} m(t_j - \tau)^{m-1} \left\{\beta\theta(\tau) + \delta\theta^3(\tau) + \varrho\theta(\tau)\gamma^2(\tau)\right\} d\tau$$

$$= \alpha(t_i) + \sum_{j=1}^{i} \int_{t_{i-1}}^{t_i} m(t_i - \tau)^{m-1} \left\{\beta\theta(\tau) + \delta\theta^3(\tau) + \varrho\theta(\tau)\gamma^2(\tau)\right\} d\tau$$

$$= \alpha(t_i) + \sum_{j=1}^{i} \{(t_i - t_{j-1})^m - (t_i - t_j)^m\} \left\{\beta\theta(\tau_j^*) + \delta\theta(\tau_j^*) + \varrho\theta(\tau_j^*)\gamma^2(\tau_j^*)\right\},$$

where we have employed the mean value theorem for integrals, which means that $t_{j-1} \leq \tau_j^* \leq t_j$, $j = 1, \ldots, i$. In a similar manner we obtain

$$(5.2) \qquad \gamma(t_i) = \mu(t_i) + \sum_{j=1}^{i} \{(t_i - t_{j-1})^m - (t_i - t_j)^m\} \cdot \left\{\zeta\gamma(\tau_j^{**}) + \eta\gamma^3(\tau_j^{**}) + \xi\gamma(\tau_j^{**})\theta^2(\tau_j^{**})\right\},$$

where $t_{j-1} \leq \tau_j^{**} \leq t_j$, $j = 1, \ldots i$.

Let

$$(5.3) \qquad U_j = \left\{\beta\theta(\tau_j^*) + \delta\theta^3(\tau_j^*) + \varrho\theta(\tau_j^*)\gamma^2(\tau_j^*)\right\} - \left\{\beta\theta(t_{j-1}) + \delta\theta^3(t_{j-1}) + \varrho\theta(t_{j-1})\gamma^2(t_{j-1})\right\}$$

and

$$(5.4) \qquad V_j = \left\{\zeta\gamma(\tau_j^{**}) + \eta\gamma^3(\tau_j^{**}) + \xi\gamma(\tau_j^{**})\theta^2(\tau_j^{**})\right\} - \left\{\zeta\gamma(t_{j-1}) + \eta\gamma^3(t_{j-1}) + \xi\gamma(t_{j-1})\theta^2(t_{j-1})\right\}.$$

Since k satisfies (4.6) and (4.8), which implies that $|\theta(t)| \leq 2C$ and $|\gamma(t)| \leq 2C$ for $0 \leq t \leq k$, we see that

$$(5.5) \qquad \begin{aligned} |U_j| &\leq \left\{\beta + 12(\delta + \varrho)C^2\right\} \omega(\theta, \gamma, h), \\ |V_j| &\leq \left\{\zeta + 12(\eta + \xi)C^2\right\} \omega(\theta, \gamma, h), \end{aligned}$$

where

(5.6) $$\omega(\theta,\gamma,h) = \max\{\omega(\theta,h),\, \omega(\gamma,h)\},$$

and

(5.7) $$\omega(\chi,h) = \sup_{\substack{0\le\tau_1<\tau_2\le k\\ |\tau_1-\tau_2|\le h}} |\chi(\tau_1)-\chi(\tau_2)|$$

is the modulus of continuity for a function χ defined on $0 \le t \le k$. Consequently,

$$\theta(t_i) = \alpha(t_i) + \sum_{j=1}^{i}\{(t_i-t_{j-1})^m - (t_i-t_j)^m\} \cdot\left\{\beta\theta(t_{j-1}) + \delta\theta^3(t_{j-1}) + \delta\theta^3(t_{j-1}) + \varrho\theta(t_{j-1})\gamma^2(t_{j-1})\right\} + \sum_{j=1}^{i}\{(t_i-t_{j-1})^m - (t_i-t_j)^m\}\,U_j,$$

(5.8) $$\gamma(t_i) = \mu(t_i) + \sum_{j=1}^{i}\{(t_i-t_{j-1})^m - (t_i-t_j)^m\} \cdot\left\{\zeta\gamma(t_{j-1}) + \eta\gamma^3(t_{j-1}) + \xi\gamma(t_{j-1})\theta^2(t_{j-1})\right\} + \sum_{j=1}^{i}\{(t_i-t_{j-1})^m - (t_i-t_j)^m\}\,V_j.$$

Now, we see that

(5.9) $$\left|\sum_{j=1}^{i}\{(t_i-t_{j-1})^m - (t_i-t_j)^m\}\,U_j\right| \le k^m\left\{\beta + 12(\delta+\varrho)C^2\right\}\omega(\theta,\gamma,h) < \omega(\theta,\gamma,h)$$

as k satisfies (4.6) and (4.8). Likewise, we find that

(5.10) $$\left|\sum_{j=1}^{i}\{(t_i-t_{j-1})^m - (t_i-t_j)^m\}\,V_j\right| \le k^m\left\{\zeta + 12(\eta+\xi)C^2\right\}\omega(\theta,\gamma,h) < \omega(\theta,\gamma,h).$$

Let u_i denote the approximations to $\theta(t_i)$, $i = 0,\ldots N$, and let v_i denote the approximations to $\gamma(t_i)$, $i = 0,\ldots N$, which are obtained from the system of recursion relations

$$
\begin{aligned}
u_0 &= \theta(0) = \alpha(0),\\
v_0 &= \gamma(0) = \mu(0),\\
u_i &= \alpha(t_i) + \sum_{j=1}^{i} \{(t_i - t_{j-1})^m - (t_i - t_j)^m\}\\
&\qquad \cdot \left\{\beta u_{j-1} + \delta u_{j-1}^3 + \varrho u_{j-1} v_{j-1}^2\right\},\\
v_i &= \mu(t_i) + \sum_{j=1}^{i} \{(t_i - t_{j-1})^m - (t_i - t_j)^m\}\\
&\qquad \cdot \left\{\zeta v_{j-1} + \eta v_{j-1}^3 + v_{j-1} u_{j-1}^2\right\}, \qquad i = 1, \ldots, N.
\end{aligned} \tag{5.11}
$$

As (5.11) is an explicit scheme, (u_i, v_i), $i = 0, \ldots, N$, is well defined. Moreover, as k satisfies (4.6) and (4.8), it can be shown as for (4.4) and (4.5) that

$$|u_i|,\ |v_i| \le 2C, \qquad i = 0, 1, \ldots, N. \tag{5.12}$$

Let

$$w_i = \theta(t_i) - u_i\,, \qquad i = 0, 1, \ldots, N, \tag{5.13}$$

and

$$z_i = \gamma(t_i) - v_i\,, \qquad i = 0, 1, \ldots, N. \tag{5.14}$$

Then

$$
\begin{aligned}
|w_0| &= 0,\\
|z_0| &= 0,\\
|w_i| &\le \omega(\theta, \gamma, h) + \sum_{j=1}^{i} \{(t_i - t_{j-1})^m - (t_i - t_j)^m\} |f_j(\theta, \gamma) - f_j(u, v)|,\\
|z_i| &\le \omega(\theta, \gamma, h) + \sum_{j=1}^{i} \{(t_i - t_{j-1})^m\\
&\qquad - (t_i - t_j)^m\} |g_j(\theta, \gamma) - g_j(u, v)|, \qquad i = 1, \ldots, N,
\end{aligned} \tag{5.15}
$$

where

$$f_j(\theta, \gamma) = \beta\theta(t_{j-1}) + \delta\theta^3(t_{j-1}) + \varrho\theta(t_{j-1})\gamma^2(t_{j-1}), \tag{5.16}$$

and

$$(5.17)\qquad g_j(\theta,\gamma) = \zeta\gamma(t_{j-1}) + \eta\gamma^3(t_{j-1}) + \xi\gamma(t_{j-1})\theta^2(t_{j-1}),$$

with the obvious substitutions for $f_j(u,v)$ and $g_j(u,v)$. Using $|\theta(t)| \le 2C$, $|\gamma(t)| \le 2C$, $|u_i| \le 2C$ and $|v_i| \le 2C$, $i = 0,\dots,N$, we see that

$$(5.18)\qquad |f_j(\theta,\gamma) - f_j(u,v)| \le k^m \left\{\beta + 12(\delta+\varrho)C^2\right\} \|(w,z)\|$$

and

$$(5.19)\qquad |g_j(\theta,\gamma) - g_j(u,v)| \le k^m \left\{\zeta + 12(\eta+\xi)C^2\right\} \|(w,z)\|,$$

where

$$(5.20)\qquad \|(w,z)\| = \max\{|w_i|, |z_i|, \quad i = 0,\dots N\}.$$

Since k satisfies (4.6) and (4.8), there exists a number $\kappa, 0 < \kappa < 1$ such that from (5.15), (5.18) and (5.19) it follows that

$$(5.21)\qquad \|(w,z)\| \le \omega(\theta,\gamma,h) + \kappa\|(w,z)\|,$$

whence

$$(5.22)\qquad \|(w,z)\| \le (1-\kappa)^{-1}\omega(\theta,\gamma,h).$$

Consequently, as $h \to 0$, $\omega(\theta,\gamma,h) \to 0$ and $\|(w,z)\| \to 0$ implying convergence of the numerical approximation to (θ,γ). We conclude this section with an estimate of $\omega(\theta,\gamma,h)$. It is easy to see that

$$(5.23)\qquad \omega(\theta,\gamma,h) < \omega(\theta,h) + \omega(\gamma,h) \le \omega(\alpha,h) + \left[\left\{2\beta C + 8(\delta+\varrho)C^3\right\} + \left\{2\zeta C + 8(\eta+\xi)C^3\right\}\right]\left\{2h^m + mht^{m-1}\right\} + \omega(\mu,h).$$

Namely, for smooth $\alpha(t)$ and $\mu(t)$ the modulus of continuity for θ and γ is $0(h^m)$. For our application here with $m = 0.125$ we use double precision arithmetic to obtain 3 to 4 significant digits.

6. The numerical study

When we consider the formulas (3.6) and (3.7) for σ_{yz} and σ_{xz} and equation (3.13) for $\theta(t)$, we see that the term

$$(6.1)\qquad G\theta(t) + \frac{1}{2}\int_0^t J(t-\tau)\theta(\tau)\,d\tau + \frac{(1+\nu)^2)}{12}\int_0^t K(t-\tau)\theta(\tau)\gamma^2(\tau)\,d\tau$$

in (3.6) and (3.7) can be replaced by the expression

$$\frac{M(t)}{I_2} - \frac{1}{4}\frac{I_3}{I_2}\int_0^t K(t-\tau)\theta^3(\tau)\,d\tau. \tag{6.2}$$

Thus the terms inside the brackets become

$$\frac{M(t)}{I_2} + \chi(x,y)\int_0^t -K(t-\tau)\theta^3(\tau)\,d\tau, \tag{6.3}$$

where

$$\chi(x,y) = \frac{I_3}{4I_2} - \frac{1}{4}\left[\left(\frac{\partial\varphi}{\partial x} - y\right)^2 + \left(\frac{\partial\varphi}{\partial y} + x\right)^2\right]. \tag{6.4}$$

Utilizing (2.7) and some elementary algebra we see that

$$\chi(x,y) = \frac{a^4(b^2 - 3y^2) + b^4(a^2 - 3x^2)}{3(a^2+b^2)^2}. \tag{6.5}$$

Formulas (6.3) through (6.5) were derived by DARWISH and CANNON in [1]. As observed there, when we consider the positive y-axis, $\chi(0,y) = 0$ for

$$y = \left(\frac{b^2(a^2+b^2)}{3a^2}\right)^{1/2} < \sqrt{\frac{2}{3}}b, \tag{6.6}$$

which means that, starting at $y = 0$ and moving up the y-axis, the absolute value of the component of stress $\sigma_{zx}(0,y,t)$ for the above nonlinear case will climb above the absolute value of the corresponding component of linear stress, and then fall below beyond $y = \sqrt{\frac{2}{3}}b$. Likewise, when we consider the positive x-axis, $\chi(0,y) = 0$ occurs when

$$x = \sqrt{\frac{1+(a/b)^2}{3}}a < a \tag{6.7}$$

or

$$\left(\frac{a}{b}\right)^2 < 2 \quad \text{or} \quad a < \sqrt{2}\,b. \tag{6.8}$$

Thus, we see that for $a < \sqrt{2}b$, starting at $x = 0$ and moving to the right along the positive x-axis, the nonlinear stress component $\sigma_{zy}(x,0,t)$ will climb above and then fall below the corresponding component of linear stress. Next, when we consider the line $y = bx/a$, where $0 \le x \le a/\sqrt{2}$, $\chi(x, bx/a) = 0$ when

$$x = a/\sqrt{3} < a/\sqrt{2}\,. \tag{6.9}$$

Thus, for increasing x values beginning at $x = 0$, the absolute values of the components σ_{xz} and σ_{yz} of the nonlinear stress will climb above the absolute values of the corresponding components of the linear stress and then fall below beyond $x = a/\sqrt{3}$.

When we consider formula (3.4) and σ_{zz} and Eq. (3.10) for $\gamma(t)$, we see that the term

$$E\frac{\gamma(t)}{2} + \frac{(1+\nu)}{2}\int_0^t J(t-\tau)\gamma(\tau)\,d\tau + \frac{(1+\nu)^3}{12}\int_0^t K(t-\tau)\gamma^3(\tau)\,d\tau \tag{6.10}$$

in expression (3.4) can be replaced by the term

$$\frac{T(t)}{I_0} - \frac{(1+\nu)}{4}\frac{I_1}{I_0}\int_0^t K(t-\tau)\gamma(\tau)\theta^2(\tau)\,d\tau. \tag{6.11}$$

Thus, the formula for the stress σ_{zz} becomes

$$\sigma_{zz}(x,y,t) = \frac{T(t)}{I_0} + d(x,y)\int_0^t -K(t-\tau)\gamma(\tau)\theta^2(\tau)\,d\tau, \tag{6.12}$$

where

$$d(x,y) = \frac{(1+\nu)}{4}\left[\frac{I_1}{I_0} - \left\{\left(\frac{\partial\varphi}{\partial y} + x\right)^2 + \left(\frac{\partial\varphi}{\partial x} - y\right)^2\right\}\right]. \tag{6.13}$$

Application of Eq. (2.7) and some elementary algebra yields

$$d(x,y) = \frac{1+\nu}{4}\left\{\frac{b^4(a^2-4x^2) + a^4(b^2-4y^2)}{(a^2+b^2)^2}\right\}. \tag{6.14}$$

When we consider the positive y-axis, $d(0,y) = 0$ for

$$y = \sqrt{\frac{b^2\left(4 + \dfrac{b^2}{a^2}\right)}{4}} < b/\sqrt{2}\,. \tag{6.15}$$

Thus, starting at $y = 0$ and moving up the y-axis, the nonlinear component σ_{zz} starts above the corresponding linear component, decreases and then falls below beyond $y = b/\sqrt{2}$. When we consider the positive x-axis, $d(x,0) = 0$ for

(6.16) $$x = \sqrt{\frac{a^2\left(\frac{a^2}{b^2}+1\right)}{4}} < a$$

when

(6.17) $$a < \sqrt{3}\,b$$

which means that if $a < \sqrt{3}b$ and x increases from $x = 0$ along the x-axis, the nonlinear component σ_{zz} of stress starts above the corresponding linear component of stress, decreases, and then falls below near $x = a$. When we consider the line $y = bx/a$, $0 \le x \le a/\sqrt{2}$, $d(x, bx/a) = 0$ for

(6.18) $$x = a/2 < a\sqrt{2}\,.$$

Thus, as $(x, bx/a)$ moves out from the origin toward the boundary of the bar, the nonlinear component of stress σ_{zz} starts above the corresponding linear component of stress, decreases and then falls below beyond the point $\left(\frac{a}{2}, \frac{b}{2}\right)$.

We perform some numerical calculations to illustrate the behaviour of the stress components and to compare the behaviour of the $\theta(t)$ and $\gamma(t)$ for the linear and nonlinear cases. In our calculations below we shall use the value

(6.19) $$\nu = 0.389$$

to complete the data set for polyurethane where we recall the definitions of $\alpha(t)$, β, δ, ϱ, $\mu(t)$, ζ, η, ξ given in (3.22), G given by (3.23), A and B given by (3.18)–(3.19), I_0 given by (3.11), I_1 given by (3.12), I_2 given by (3.14), and I_3 given by (3.15). Recall that we are interested in the behaviour when the tension

$$T(t) = \begin{cases} 0, & 0 \le t \le t_1, \\ T, & t_1 \le t. \end{cases}$$

The selection of the numerical procedure was made with this case in mind since the computations will yield $\gamma(t_i) = 0$ as long as $T(t_i) = 0$. We note also that the solution $(\theta(t), \gamma(t))$ exists and is unique, what can be shown by applying the above results for $0 \le t \le t_1$ and then applying the above results for $t_1 \le t$. This will result in a jump for $\gamma(t)$ while $\theta(t)$ remains continuous. See CANNON [9] for the solution of similar problems with piecewise continuous data.

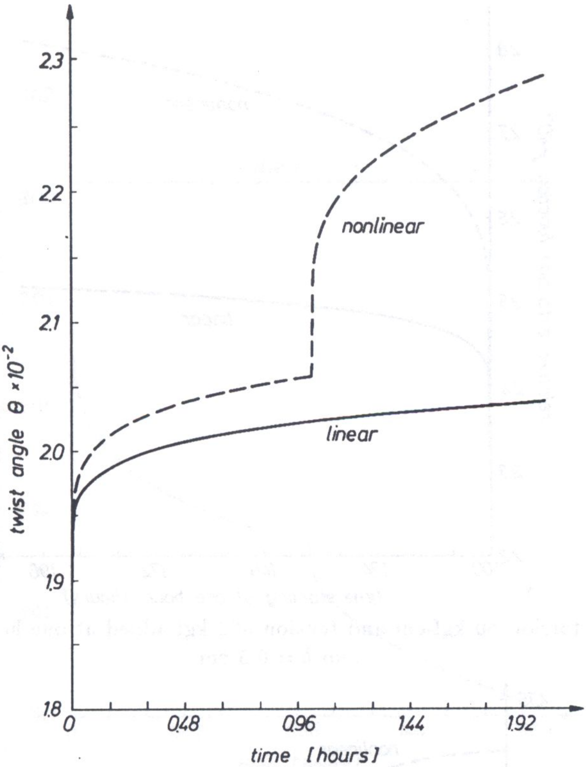

FIG. 2. θ for torsion 50 kgf-cm and tension 500 kgf added at one hour; $a = 0.75$ cm and $b = 0.5$ cm.

We considered the case of $a = 0.75$ cm and $b = 0.5$ cm with an initial constant torsion of 50 kg/cm and tension 500 kg added after one hour. We calculated θ and γ up to $k = 2$ hours using $N = 1000$ and $h = 0.002$. The behaviour of θ is shown in Fig. 2. Note the influence of the tension applied at one hour for the nonlinear case. The behaviour of γ is shown in Fig. 3. Note that γ is identically zero for the first hour and jumps at $t = 1$ with the application of the tension. Note that γ for the nonlinear case climbs above γ for the corresponding linear case. In Fig. 4, we display the graph of the normal stress $\sigma_{zz}(0, y, 2)$ as a function of y to illustrate the effect of the function $d(x, y)$ in (6.13). Note that the normal stress σ_{zz} for the nonlinear case starts above the σ_{zz} for the corresponding linear case at $y = 0$, then it decreases, and falls below the linear case as y tends to b. In Fig. 5, we display the effect of the applied tension on the shearing stress $\sigma_{yz}(a/\sqrt{2}, b/\sqrt{2}, t)$ as a function of time. We note that as a function of time, $\sigma_{yz}(a/\sqrt{2}, b/\sqrt{2}, t)$ for the nonlinear case drops away from the linear case value. Note the impact of the application of tension at $t = 1$ hour which changes abruptly the

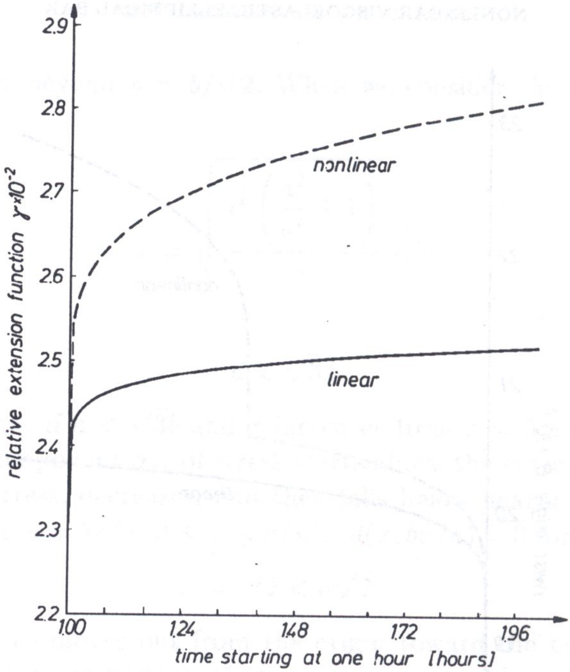

FIG. 3. γ for torsion 50 kgf-cm and tension 500 kgf added at one hour; $a = 0.75$ cm and $b = 0.5$ cm.

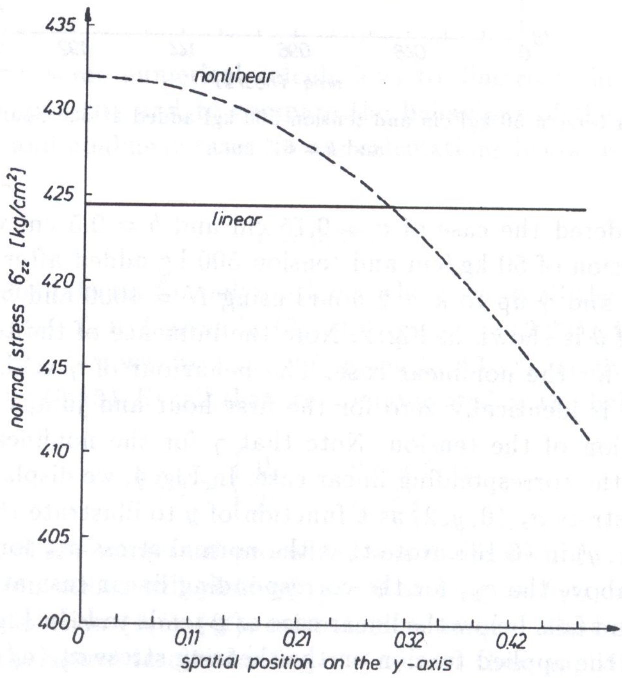

FIG. 4. Stress σ_{zz} at two hours on the y-axis for torsion 50 kgf-cm and tension 500 kgf added at one hour; $a = 0.75$ cm and $b = 0.5$ cm.

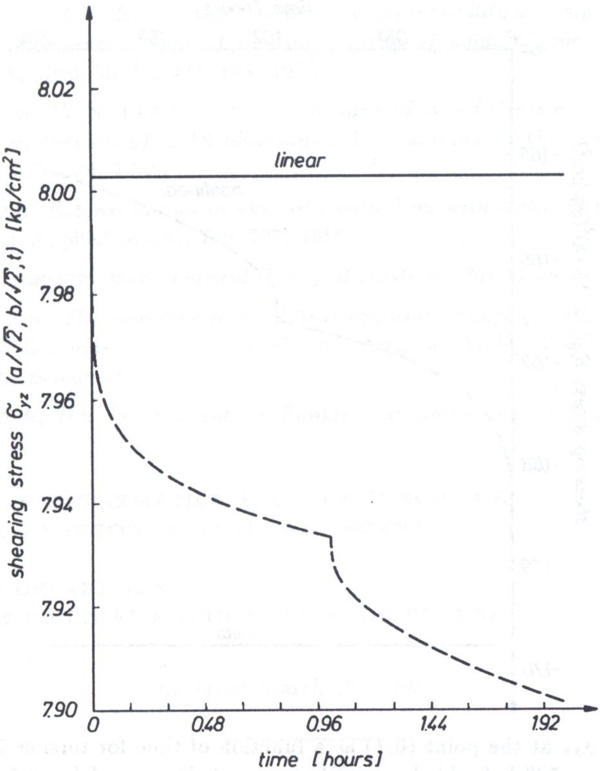

FIG. 5. Stress σ_{yz} at the point $(a/\sqrt{2}, b/\sqrt{2})$ as a function of time for torsion 50 kgf-cm and tension 500 kgf added at one hour; $a = 0.75$ cm and $b = 0.5$ cm.

derivative of the nonlinear $\sigma_{yz}(a/\sqrt{2}, b/\sqrt{2}, t)$ near $t = 1^+$. The curve rapidly settles down to a slightly larger rate of decrease over the nonlinear σ_{yz} with no application of tension. Note the similar behaviour of the shearing stress $\sigma_{xz}(0, b, t)$ displayed in Fig. 6.

To summarize, we note that

1. The relative angle of twist function $\theta(t)$ increases more rapidly when torsion acts together with tension than when torsion acts alone.

2. We note that for constant tension, the relative extension function $\gamma(t)$ increases with time.

3. From the system of Eqs. (3.20) and (3.21), it is clear that for a fixed torsion couple, the relative extension $\gamma(t)$ increases as the constant value of tension T increases.

4. The results 1, 2, and 3 above are not evident in the linear theory.

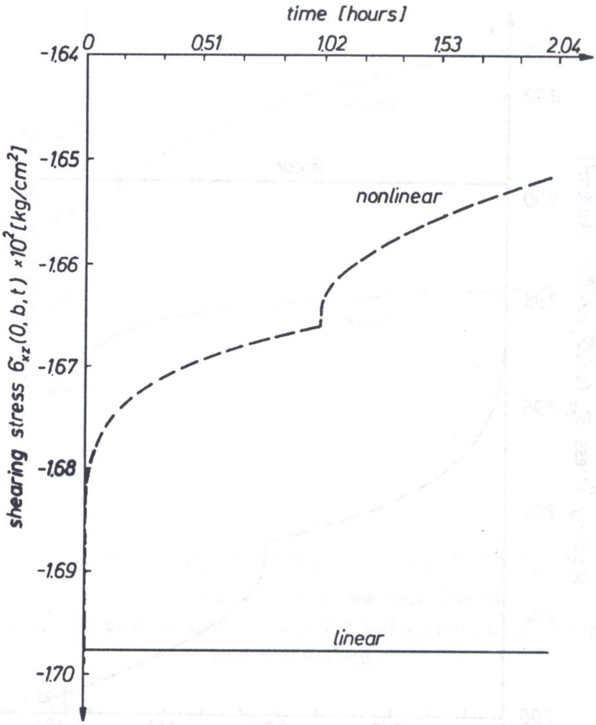

FIG. 6. Stress σ_{xz} at the point $(0, b)$ as a function of time for torsion 50 kgf-cm and tension 500 kgf added at one hour; $a = 0.75$ cm and $b = 0.5$ cm.

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