ENGINEERING TRANSACTIONS • Engng. Trans. • 45, 3–4, 463–470, 1997
Polish Academy of Sciences • Institute of Fundamental Technological Research

APPROXIMATE SOLUTIONS OF DYNAMIC PROBLEMS FOR RECTANGULAR COMPOSITE PLATES

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We study the problem of dynamic deformation of rectangular composite plates with the same type of boundary conditions along each side. Generalized theory of plates taking into account the compliance to transversal shear strains is used. The building method of an approximate analytical solution is proposed. We have obtained the numerical result for the case of the impact load acting on the rigidly fixed plate.

1. Introduction

Deformation and stresses produced by impact loading in anisotropic composite plates are analyzed. The set of differential equations governing the behaviour of the plate model assumed cannot be solved to yield closed-form solutions, contrary to the classical case discussed in [1]. All edges of the plate are assumed to be supported in the same manner, four types of boundary conditions are discussed.

2. Formulation of the problem

Consider a rectangular orthotropic plate of thickness $2h$ and dimensions $2a \times 2b$ referred to a rectangular coordinate system, with origin at the center of the plate. The plate is subject to dynamic loading. Following [1], the inertia terms corresponding to horizontal motions of the plate elements are disregarded. Under such assumptions, the equations of motion of the plate and the corresponding generalized force-displacement relations take the following form (cf. [2]):

$$\frac{\partial^2 w}{\partial x^2} + \frac{1}{g}\frac{\partial^2 w}{\partial y^2} + \frac{\partial \gamma_1}{\partial x} + \frac{1}{g}\frac{\partial \gamma_2}{\partial y} = \frac{1}{c^2}\left[\frac{\partial^2 w}{\partial t^2} - a(t)\right],$$

$$(2.1)\qquad \frac{\partial^2 \gamma_1}{\partial x^2} + e_1\frac{\partial^2 \gamma_1}{\partial y^2} - \kappa_1^2\gamma_1 + (e_1+\nu_2)\frac{\partial^2 \gamma_2}{\partial x \partial y} - \kappa_1^2\frac{\partial w}{\partial x} = 0,$$

$$\frac{\partial^2 \gamma_2}{\partial y^2} + e_2\frac{\partial^2 \gamma_2}{\partial x^2} - \kappa_2^2\gamma_2 + (e_2+\nu_1)\frac{\partial^2 \gamma_1}{\partial x \partial y} - \kappa_2^2\frac{\partial w}{\partial y} = 0.$$

The elasticity relations between the generalized displacements and stresses are as follows:

$$M_1 = D_1\left(\frac{\partial\gamma_1}{\partial x} + \nu_2\frac{\partial\gamma_2}{\partial y}\right), \qquad M_2 = D_2\left(\frac{\partial\gamma_2}{\partial y} + \nu_1\frac{\partial\gamma_1}{\partial x}\right),$$

(2.2)
$$H = D_{12}\left(\frac{\partial\gamma_1}{\partial y} + \frac{\partial\gamma_2}{\partial x}\right),$$

$$Q_1 = \Lambda_1\left(\gamma_1 + \frac{\partial w}{\partial x}\right), \qquad Q_2 = \Lambda_2\left(\gamma_2 + \frac{\partial w}{\partial y}\right).$$

In these formulae

$$g = \frac{G_{13}}{G_{23}}, \quad c^2 = \frac{6}{5\varrho}G_{13}, \quad e_i = \frac{G_{12}}{E_i(1-\nu_1\nu_2)}, \quad \kappa_i^2 = \frac{5}{2}\frac{G_{i3}(1-\nu_1\nu_2)}{E_i h^2},$$

ν_i is Poisson's ratio; E_i are Young's moduli; G_{12}, G_{13}, G_{23} are the shear moduli;

$$\Lambda_i = \frac{28h}{15}G_{i3}, \qquad D_i = \frac{2h^2}{3}E_i, \qquad D_{12} = \frac{2h^2}{3}G_{12}, \qquad i = 1,2.$$

Both the initial and boundary conditions at $t = t_0$ and $x = \pm a$, $y = \pm b$ should be added to make the solution unique.

For uniform initial conditions under the dynamic loading we have

(2.3)
$$w(x,y,t)\Big|_{t=t_0} = 0, \qquad \frac{\partial}{\partial t}w(x,y,t)\Big|_{t=t_0} = 0.$$

Consider the most typical methods of modelling the plate edge supports and present them in terms of the generalized displacements and stresses:

a) clamped edge:

(2.4)
$$\gamma_i(x,y,t)\Big|_{x=\pm a} = \gamma_i(x,y,t)\Big|_{y=\pm b} = 0, \qquad i = 1,2,$$
$$w(x,y,t)\Big|_{x=\pm a} = w(x,y,t)\Big|_{y=\pm b} = 0;$$

b) simply supported edge:

(2.5)
$$w(x,y,t)\Big|_{x=\pm a} = w(x,y,t)\Big|_{y=\pm b} = 0,$$
$$M_1(x,y,t)\Big|_{x=\pm a} = 0, \qquad M_2(x,y,t)\Big|_{y=\pm b} = 0,$$
$$H(x,y,t)\Big|_{x=\pm a} = H(x,y,t)\Big|_{y=\pm b} = 0;$$

c) elastic support:

$$
\begin{aligned}
& w(x,y,t)\Big|_{x=\pm a} = w(x,y,t)\Big|_{y=\pm b} = 0, \\
& [M_1(x,y,t) - k_1\gamma_1(x,y,t)]\Big|_{x=\pm a} = 0, \\
(2.6)\qquad & [M_2(x,y,t) - k_1\gamma_2(x,y,t)]\Big|_{y=\pm b} = 0, \\
& [H(x,y,t) - k_2\gamma_1(x,y,t)]\Big|_{y=\pm b} = 0, \\
& [H(x,y,t) - k_2\gamma_2(x,y,t)]\Big|_{x=\pm a} = 0,
\end{aligned}
$$

k_1, k_2 – are the coefficients of support rigidity;

d) free edge

$$
\begin{aligned}
& M_1(x,y,t)\Big|_{x=\pm a} = M_2(x,y,t)\Big|_{y=\pm b} = 0, \\
(2.7)\qquad & H(x,y,t)\Big|_{x=\pm a} = H(x,y,t)\Big|_{y=\pm b} = 0, \\
& Q(x,y,t)\Big|_{x=\pm a} = Q(x,y,t)\Big|_{y=\pm b} = 0.
\end{aligned}
$$

3. Solution

The boundary conditions (2.4) are satisfied along the plate boudary if the solution of the set of Eqs. (2.1) is written in the form

$$
\begin{aligned}
& w(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} w_{nm}(t)\cos\alpha_n x\cos\beta_m y, \\
& \gamma_1(x,y,t) = \left(\frac{x^3}{a^3} - \frac{x}{a}\right)\sum_{m=0}^{\infty}\gamma_m^{(1)}(t)\cos\beta_m y, \\
(3.1)\qquad & \gamma_2(x,y,t) = \left(\frac{y^3}{b^3} - \frac{y}{b}\right)\sum_{n=0}^{\infty}\gamma_n^{(2)}(t)\cos\alpha_n x, \\
& \alpha_n = \frac{2n+1}{2a}\pi, \qquad \beta_m = \frac{2m+1}{2b}\pi.
\end{aligned}
$$

Substituting the relations (3.1) into Eqs. (2.1) we obtain

$$
(3.2)\qquad -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} w_{nm}(t)\left(\alpha_n^2 + \frac{1}{g}\beta_m^2\right)\cos\alpha_n x\cos\beta_m y
$$

$$
+ \frac{1}{a}\left(\frac{3x^2}{a^2} - 1\right)\sum_{m=0}^{\infty}\gamma_m^{(1)}(t)\cos\beta_m y
$$

(3.2) [cont.]

$$+\ \frac{1}{gb}\left(\frac{3y^2}{b^2}-1\right)\sum_{n=0}^{\infty}\gamma_n^{(2)}(t)\cos\alpha_n x$$

$$=\frac{1}{c^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[\ddot{w}_{nm}(t)-a(t)\delta_{nm}]\cos\alpha_n x\cos\beta_m y,$$

$$\kappa_1^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\alpha_{nm}w_{nm}(t)\sin\alpha_n x\cos\beta_m y$$

$$-\frac{e_1+\nu_2}{b}\left(\frac{3y^2}{b^2}-1\right)\sum_{n=0}^{\infty}\alpha_n\gamma_n^{(2)}(t)\sin\alpha_n x$$

$$+\sum_{m=0}^{\infty}\left[\frac{6x}{a^3}-\left(e_1\beta_m^2+\kappa_1^2\right)\left(\frac{x^3}{a^3}-\frac{x}{a}\right)\right]\gamma_m^{(1)}(t)\cos\beta_m y=0,$$

$$\kappa_2^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\beta_m w_{nm}(t)\cos\alpha_n x\sin\beta_m y$$

$$-\frac{e_2+\nu_1}{a}\left(\frac{3x^2}{a^2}-1\right)\sum_{m=0}^{\infty}\beta_m\gamma_m^{(1)}(t)\sin\beta_m y$$

$$+\sum_{m=0}^{\infty}\left[\frac{6y}{b^3}-\left(e_2\alpha_n^2+\kappa_2^2\right)\left(\frac{y^3}{b^3}-\frac{y}{b}\right)\right]\gamma_n^{(2)}(t)\cos\alpha_n x=0.$$

Then the Bubnov- Galerkin method [3] is used in variables x and y, choosing as the coordinate functions the expression $\{\cos\alpha_n x\cos\beta_m y\}_{n,m=0}^{\infty}$ for the first equation, $\{\sin\alpha_n x\cos\beta_m y\}_{n,m=0}^{\infty}$ for the second equation, and $\{\cos\alpha_n x\sin\beta_m y\}_{n,m=0}^{\infty}$ for the third equation. After integration we obtain the expressions for the coefficients $\gamma_m^{(1)}(t)$, $\gamma_n^{(2)}(t)$ in terms of $w_{nm}(t)$:

$$\gamma_m^{(1)}(t)=\frac{\Delta_1}{\Delta}w_{nm}(t),\qquad \gamma_n^{(2)}(t)=\frac{\Delta_2}{\Delta}w_{nm}(t), \tag{3.3}$$

where

$$\Delta=a_{11}a_{22}-a_{12}a_{21},\quad \Delta_1=\kappa_2^2\beta_m a_{12}-\kappa_1^2\alpha_n a_{22},\quad \Delta_2=\kappa_1^2\alpha_n a_{21}-\kappa_2^2\beta_m a_{11},$$

$$a_{11}=\frac{48}{a^2}\delta_n^{(1)}-\frac{2}{a}\left(e_1\beta_m^2+\kappa_1^2\right)d_n^{(1)}/\alpha_n,\qquad a_{22}=\frac{48}{b^2}\delta_m^{(2)}-\frac{2}{b}\left(e_2\alpha_n^2+\kappa_2^2\right)d_m^{(2)}/\beta_m,$$

$$a_{12}=-2\frac{e_1+\nu_2}{b}\alpha_n d_m^{(2)},\qquad a_{21}=-2\frac{e_2+\nu_1}{a}\beta_m d_n^{(1)},$$

$$d_n^{(1)}=\frac{4(-1)^n}{(2n+1)\pi}\left[1-\frac{12}{(2n+1)^2\pi^2}\right],\qquad d_m^{(2)}=\frac{4(-1)^m}{(2m+1)\pi}\left[1-\frac{12}{(2m+1)^2\pi^2}\right],$$

$$\delta_n^{(1)}=\frac{(-1)^n}{(2n+1)^2\pi^2},\qquad \delta_m^{(2)}=\frac{(-1)^m}{(2m+1)^2\pi^2}.$$

Functions $w_{nm}(t)$ are found from the equation

$$\ddot{w}_{nm}(t) + \mu_{nm}^2 w_{nm}(t) = \delta_{nm} a(t), \tag{3.4}$$

where

$$\delta_{nm} = \frac{16(-1)^n(-1)^m}{(2n+1)(2m+1)\pi^2},$$
$$\mu_{nm}^2 = c^2\left[\alpha_n^2 + \frac{1}{g}\beta_m^2 - \frac{2}{a}d_n^{(1)}\frac{\Delta_1}{\Delta} - \frac{2}{b}d_m^{(2)}\frac{\Delta_2}{\Delta}\right].$$

The solution of Eq. (3.4) under uniform initial conditions, is of the following form:

$$w_{nm}(t) = \frac{\delta_{nm}}{\mu_{nm}}\int_0^t a(\xi)\sin\mu_{nm}(t-\xi)\,d\xi. \tag{3.5}$$

If the dynamic loading (impact) is given in the form

$$a(t) = \begin{cases} a_0 & \text{at } 0 \le t \le \tau, \\ 0 & \text{at } t \ge \tau, \end{cases}$$

where τ is the duration of the load impulse, a_0 is its intensity, then the expression (3.5) will be of the form

$$w_{nm}(t) = \frac{\delta_{nm} a_0}{\mu_{nm}^2} f_{nm}, \tag{3.6}$$

where

$$f_{nm} = \begin{cases} 1 - \cos\mu_{nm}t & \text{at } 0 \le t \le \tau, \\ 2\sin\mu_{nm}(t-\tau/2)\sin\mu_{nm}\tau/2 & \text{at } t \ge \tau. \end{cases}$$

Thus, for a plate rigidly fixed at its edges, we have obtained a closed solution allowing to consider the plate response to the transverse impact loading.

Using the combinations of trigonometrical functions and power series, the boundary conditions (2.5), (2.6), (2.7) can be also satisfied.

4. Example

As an example, a square plate made of high-quality polymer material – propylene-ethylene copolymer – is considered. For a plate with rigidly clamped edges the parameters are as follows:

$$a = 0.34\,\text{m}, \qquad h = 6\cdot 10^{-3}\,\text{m}, \qquad \varrho = 0.91\cdot 10^3\,\text{kg/m}^3,$$
$$E = 1.2\cdot 10^3\,\text{MPa} \qquad \nu = 0.412.$$

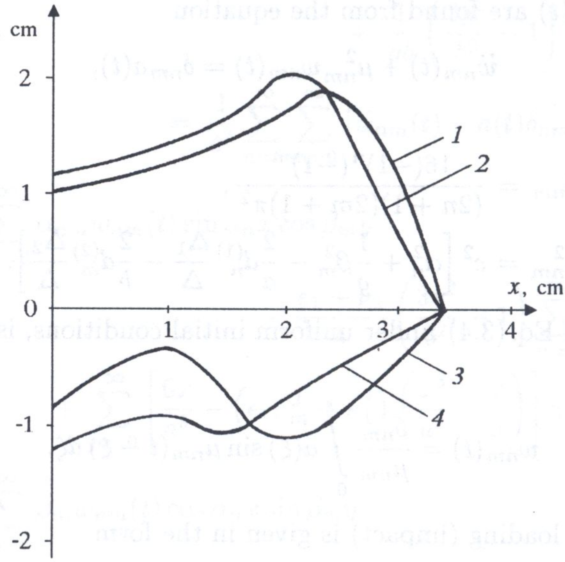

FIG. 1. Deflection along the Ox axis at $y = 0$ for several values of time $\tau = 0.001$ s: curve *1* at $t = 100\tau$, curve *2* at $t = 150\tau$, curve *3* at $t = 300\tau$, curve *4* at $t = 500\tau$.

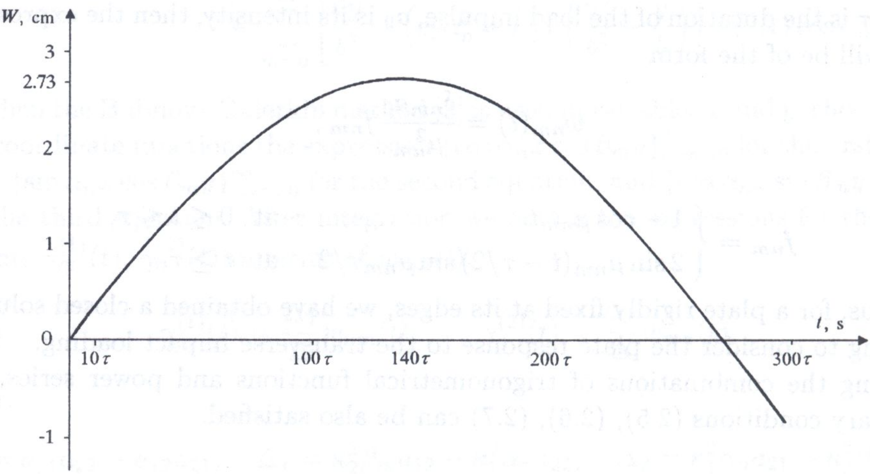

FIG. 2. Variation of deflection in time ($\tau = 0.001$ s) in the centre of the plate (at point $(0, 0)$).

The amplitude of the impact pulse is $a_0 = 196.2\,\mathrm{m/s^2}$, and the duration time $\tau = 0.001$ s.

In Fig. 1 we have plotted the deflection along the axis Ox at $y = 0$ for several values of time. The variation of the deflection in time at the centre of the plate is shown in Fig. 2. In Fig. 3 is shown the variation of stress σ_{11} in time at several points for the plate considered. The diagram of the stress σ_{11} at the same points

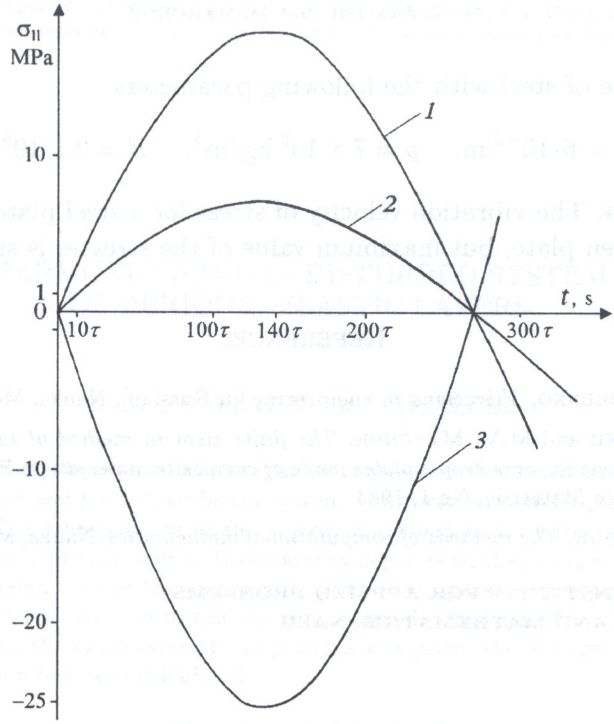

FIG. 3. Variation of stress σ_{11} (for polymetric material) by time ($\tau = 0.001$ s) at several points: curve *1* at (0,0), curve *2* at $(0, a/2)$, curve *3* at $(0, a)$.

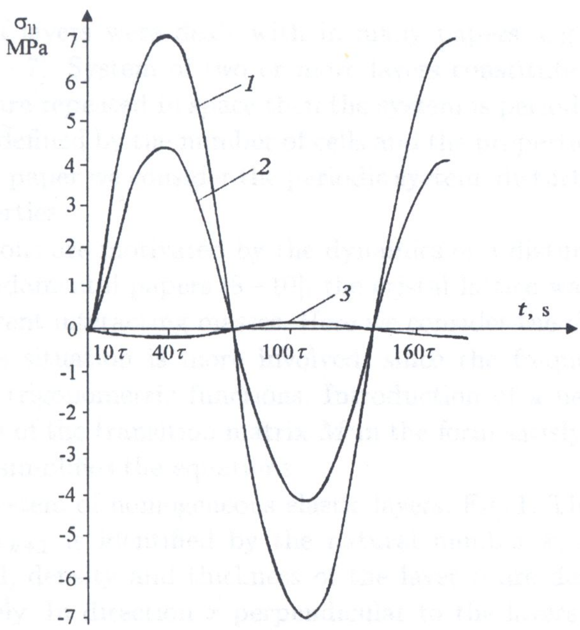

FIG. 4. Variation of stress σ_{11} (for steel material) by time ($\tau = 0.001$ s) at several points: curve *1* at (0,0), curve *2* at $(0, a/2)$, curve *3* at $(0, a)$.

for a plate made of steel with the following parameters:

$$a = 0.34\,\mathrm{m}, \quad h = 6\cdot 10^{-3}\,\mathrm{m}, \quad \varrho = 7.8\cdot 10^{3}\,\mathrm{kg/m^3}, \quad E = 2.1\cdot 10^{5}\,\mathrm{MPa}, \quad \nu = 0.3$$

is given in Fig. 4. The vibration velocity of stress for a steel plate is greater than that for the given plate, but maximum value of the stresses is smaller.

References

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Received October 7, 1997.