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A SIMPLIFIED MODEL OF THE KIRCHHOFF PLATE RESTING ON THE ELASTIC SUBSOIL

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An approximate 2D-model is proposed for the analysis of thin plates resting on a two-parameter subsoil layer. The model obtained is discretized and applied to calculations of uniformly loaded square plates. The scope of applicability of the analytical formulae for the plate deflections and curvatures is evaluated.

1. Introduction

The aim of this contribution is to propose and apply a simplified 2D-model of a Kirchhoff plate on an elastic subsoil layer. The layer is resting on a rigid substratum and is represented by means of the Vlasov model. Problems of this kind have been analyzed in a series of monographs, textbooks and papers, cf. [1, 2, 4], the overview of which can be found in [3]. Nevertheless, in order to obtain solutions to various engineering problems, extensive calculations are necessary. The main feature of the proposed model is that after discretization, it leads to a system of algebraic equations with a symmetric matrix. The model was applied to the evaluation of deflections and curvatures of uniformly loaded square plates. The results are presented in the form of simple algebraic formulas and illustrated by diagrams.

2. Preliminaries

The scheme of a plate interacting with a subsoil layer is shown in Fig. 1. The displacements of an arbitrary point of this system are denoted by u_α, u_3; here and in the sequel all Greek subscripts assume the values 1, 2 corresponding to Cartesian coordinates x_1, x_2. The subsoil displacements are assumed in the form

$$u_3(\mathbf{x}, z) = w(\mathbf{x})\psi(z), \qquad u_\alpha(\mathbf{x}, z) = 0,$$

where $\psi(\cdot)$ is a decreasing monotone function satisfying conditions $\psi(0) = 1$,

$\psi(H) = 0$, cf. [4]. At the same time, stresses in the subsoil are given by (cf. [4], p. 51)

$$\sigma_{33} = \frac{E_0}{1-\nu_0^2}\psi_{,3}(z)w(\mathbf{x}), \qquad \sigma_{3\alpha} = \frac{E_0}{2(1+\nu_0)}\psi(z)w_{,\alpha}(\mathbf{x}),$$

with

$$E_0 \equiv \frac{E_s}{1-\nu_s^2}, \qquad \nu_0 \equiv \frac{\nu_s}{1-\nu_s},$$

where E_s, ν_s are Young's modulus and Poisson's ratio, respectively, related to the subsoil. The above formulae hold for $\mathbf{x} \in R^2$, $z \in (0, H)$. The plate displacements are assumed in the known form

$$u_3(\mathbf{x}, \zeta) = w(\mathbf{x}), \qquad u_\alpha(\mathbf{x}, \zeta) = -\zeta w_{,\alpha}(\mathbf{x}),$$

where $\zeta \equiv z + h \in (-h, h)$, $\mathbf{x} \in \Omega$. At the same time the plate material is assumed to have elastic symmetry planes $\zeta = \text{const}$, and stresses in the plate are given by

$$\sigma_{\alpha\beta} \equiv D_{\alpha\beta\gamma\delta}u_{(\gamma,\delta)}, \qquad D_{\alpha\beta\gamma\delta} \equiv C_{\alpha\beta\gamma\delta} - \frac{C_{\alpha\beta33}C_{\gamma\delta33}}{C_{3333}},$$

where $C_{\alpha\beta\gamma\delta}$, $C_{\alpha\beta33}$, C_{3333} are components of elastic moduli tensor of the plate material which are even functions of $\zeta \in (-h, h)$.

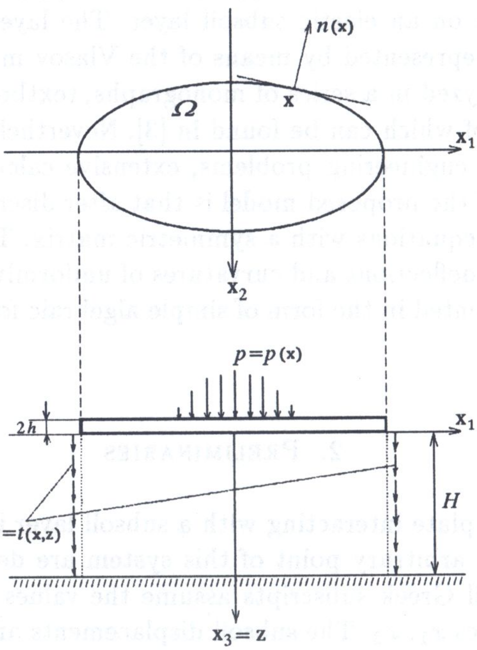

FIG. 1.

Let $t = t(\mathbf{x}, z)$, $\mathbf{x} \in \partial\Omega$, $z \in (0, H)$ be the tangent tractions acting along the z-axis on a part of the subsoil under the plate, cf. Fig. 1. Then the weak form of the equilibrium equation for a plate-subsoil system under consideration is given by

$$(2.1)\qquad G_{\alpha\beta\gamma\delta}\int_{\Omega} w_{,\gamma\delta}\,\delta w_{,\alpha\beta}\,dA + s\int_{\Omega} w\delta w\,dA + r\int_{\Omega} w_{,\alpha}\delta w_{,\alpha}\,dA = \oint_{\partial\Omega}\int_0^H t\psi\,dz\,\delta w\,dS + \int_{\Omega} p\delta w\,dA$$

and

$$(2.2)\qquad s\int_{R^2\backslash\Omega} w\delta w\,dA + r\int_{R^2\backslash\Omega} w_{,\alpha}\delta w_{,\alpha}\,dA = \oint_{\partial\Omega}\int_0^H \bar{t}\psi\,dz\,\delta w\,dS,$$

where

$$G_{\alpha\beta\gamma\delta} \equiv \int_{-h}^{h} D_{\alpha\beta\gamma\delta}(\zeta)^2\,d\zeta, \qquad s \equiv \frac{E_0}{1-\nu_0^2}\int_0^H (\psi_{,3})^2\,d\zeta, \qquad r \equiv \frac{E_0}{2(1+\nu_0)}\int_0^H (\psi)^2\,d\zeta,$$

and $\bar{t}$ are tangent tractions acting along the z-axis on a part of a subsoil layer situated outside the plate. The above weak form of equilibrium equations has to hold for an arbitrary sufficiently regular function δw such that $\delta w \to 0$ together with all derivatives if $x_1 \to \pm\infty$ or $x_2 \to \pm\infty$. The general 2D-model of the plate-subsoil system is obtained by eliminating t, $\bar{t}$ from the above equations by means of $t + \bar{t} = 0$. Application of this model leads to rather troublesome calculations and will be replaced below by a simplified model proposed in this paper. For the rectangular plate the model proposed is similar to that applied in [4], pp. 194–203.

3. Simplified 2D-model

Let us assume that Eq. (2.2) holds for an arbitrary sufficiently regular δw. Then

$$(3.1)\qquad \begin{gathered} w_{,\alpha\alpha} - \gamma^2 w = 0 \qquad \text{in } R^2\backslash\Omega, \\ \gamma^2 \equiv \frac{s}{r}, \qquad r w_{,\alpha} n_\alpha = -\int_0^H \bar{t}\psi\,dz \qquad \text{on } \partial\Omega, \end{gathered}$$

where n_α is a unit normal outward to Ω at $\partial\Omega$. In the framework of the proposed simplified model, the continuity conditions for $u_{3,\alpha} = w_{,\alpha}(\mathbf{x})\psi(z)$ across $\partial\Omega \times$

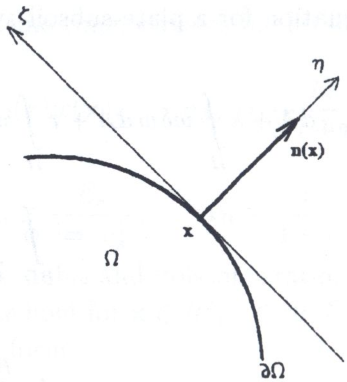

FIG. 2.

$(0, H)$ will be not satisfied. Hence $t + \bar{t} \neq 0$. Let $w = w_x(\eta, \zeta)$ in the vicinity of $\mathbf{x} \in \partial\Omega$, where η, ζ are shown in Fig. 2. The basic assumption leading to the simplified 2D-model of a plate on an elastic subsoil is that in the vicinity of every $\mathbf{x} \in \partial\Omega$ in region $R^2 \backslash \Omega$, derivatives $\partial^2 w_{\mathbf{x}} / \partial\zeta^2$ can be neglected as compared to $\partial^2 w_{\mathbf{x}} / \partial\eta^2$. In this case we obtain from (3.1) that $w_{\mathbf{x}}(\eta, 0) = w_{\mathbf{x}} \exp(-\gamma\eta)$, $\mathbf{x} \in \partial\Omega$. Hence

$$u_3(\mathbf{x} + \eta\mathbf{n}(\mathbf{x}), z) = w(\mathbf{x}) \exp(-\gamma\eta)\psi(z), \qquad \eta \geq 0,$$

and

$$(3.2) \qquad t(\mathbf{x}, z) = \frac{E_0}{(1+\nu_0)} \frac{1}{2} \frac{\partial \mathbf{u}_3}{\partial \eta} = -\frac{1}{2} \frac{E_0}{(1+\nu_0)} \gamma w(\mathbf{x}) \psi(z), \qquad z \in (0, H), \quad \mathbf{x} \in \partial\Omega.$$

The introduced modeling approximation leads to the discontinuity of derivatives of $w(\cdot)$ across the $\partial\Omega$, and with notation

$$m \equiv \frac{E_0 \gamma}{2(1+\nu_0)} \int_0^H (\psi)^2 \, dz = r\gamma = \sqrt{rs}\,,$$

gives rise to the residuals on $\partial\Omega$ defined by

$$(3.3) \qquad S \equiv -\int_0^H t\psi \, dz - \int_0^H \bar{t}\psi \, dz = mw(\mathbf{x}) + rw_{,\alpha}(\mathbf{x}) n_\alpha(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega.$$

Substituting the RHS of Eqs. (3.2) into (2.1) we obtain

$$(3.4) \qquad G_{\alpha\beta\gamma\delta} \int_\Omega w_{,\gamma\delta} \delta w_{,\alpha\beta} \, dA + s \int_\Omega w \, \delta w \, dA + r \int_\Omega w_{,\alpha} \delta w_{,\alpha} \, dA + m \oint_\Omega w \delta w \, dS = \int_\Omega p \, \delta w \, dA,$$

for an arbitrary, sufficiently regular δw satisfying the possible constraints imposed on the plate deflections $w(\mathbf{x}), \mathbf{x} \in \Omega$. Setting

$$
\begin{aligned}
R &\equiv \frac{G_{\alpha\beta\gamma\delta}}{s} w_{,\alpha\beta\gamma\delta} - \frac{p}{s} + \left(w - \frac{r}{s} w_{,\alpha\alpha}\right) \quad \text{in } \Omega, \\
M_\alpha &\equiv \frac{G_{\alpha\beta\gamma\delta}}{s} w_{,\gamma\delta} n_\beta \quad \text{on } \partial\Omega, \\
Q &\equiv \left(-\frac{G_{\alpha\beta\gamma\delta}}{s} w_{,\gamma\delta\beta} + \frac{r}{s} w_{,\alpha}\right) n_\alpha + \frac{m}{s} w \\
&= -\frac{G_{\alpha\beta\gamma\delta}}{s} w_{,\gamma\delta\beta} n_\alpha + \frac{S}{s} \quad \text{on } \partial\Omega,
\end{aligned} \tag{3.5}
$$

we can write Eq. (3.4) in the form

$$\int_\Omega R\delta w\, dA + \oint_{\partial\Omega} M_\alpha \delta w_{,\alpha}\, dS + \oint_{\partial\Omega} Q\delta w\, dS = 0.$$

Denoting by η, ζ the parameters related to the local directions at $\mathbf{x}$, $\mathbf{x} \in \partial\Omega$, shown in Fig. 2, the above condition yields

$$\int_\Omega R\delta w\, dA + \oint_{\partial\Omega} (Q - M_{\zeta,\zeta})\delta w\, dS + \oint_{\partial\Omega} M_\eta \delta w_{,\eta}\, dS = 0,$$

where $M_\eta \equiv M_\alpha n_\alpha$, $M_\zeta \equiv M_\alpha \in_{\beta\alpha} n_\beta$, $\in_{\beta\alpha}$ being the permutation symbol. The terms in brackets in Eqs. $(3.5)_1$ define interactions between the plate and the subsoil. Equation (3.4) represents the proposed simplified 2D-model of the plate resting on the subsoil layer. The total strain energy of this model is given by the functional

$$\mathcal{W}(w) = \frac{1}{2}\left[\int_\Omega \left(G_{\alpha\beta\gamma\delta} w_{,\alpha\beta} w_{,\gamma\delta} + r w_{,\alpha} w_{,\alpha} + s w^2\right) dA + \oint_{\partial\Omega} w^2\, dS\right]. \tag{3.6}$$

REMARK. Setting $S = 0$ in (3.3) and using the second of Eqs. (3.1), we can eliminate the first term on the RHS of Eq. (2.1). This procedure also leads to an approximate 2D-model of the system under considerations. However, this model has a certain unreasonable feature due to the fact that the total strain energy of the whole system does not exist. Moreover, the postulated assumption $S = 0$ is not consistent with the procedure based on the concept of constraints, where the discontinuity of $w_{,\alpha} n_\alpha$ across $\partial\Omega$ has to be maintained by the reaction forces S defined by (3.3). If $R = 0$, $Q - M_{\zeta,\zeta} = 0$, $M_\eta = 0$ for some solution $w(\cdot)$ to Eq. (3.4), then this solution (in a framework of a simplified 2D-model) will be called exact.

Let L be the smallest characteristic length dimension of the region Ω. Setting $\delta = H/L$ we observe that $(m/s) \in \mathcal{O}(\delta)L$, $(r/s) \in \mathcal{O}(\delta^2)L^2$. Assuming that $\delta \to 0$, we can pass to the asymptotic approximation of the proposed simplified 2D-model. This approximation can be used provided that $H/L \ll 1$. In this case in Eqs. (3.4) and Eqs. (3.5) the terms involving r and m can be neglected, and the residuals will be given by

$$\begin{aligned} \overline{R} &\equiv G_{\alpha\beta\gamma\delta} w_{,\alpha\beta\gamma\delta} + sw - p \qquad \text{in } \Omega, \\ \overline{M}_\alpha &\equiv G_{\alpha\beta\gamma\delta} w_{,\gamma\delta} n_\beta \qquad \text{on } \partial\Omega, \\ \overline{Q} &\equiv -G_{\alpha\beta\gamma\delta} w_{,\beta\gamma\delta} n_\alpha \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.7}$$

A solution $w(\cdot)$ obtained in the framework of the above asymptotic approximation will be called exact if $\overline{R} = 0$, $\overline{M}_\eta = 0$, $\overline{Q} - \overline{M}_{\zeta,\zeta} = 0$. It can be seen that this approximation represents the well known 2D-model of a plate resting on the one-parameter elastic subsoil.

An example of application of the introduced 2D-model of a plate resting on a subsoil layer, given by Eq. (3.4), will be presented in the subsequent sections.

4. Discretized model

Let us look for a solution $w(\cdot)$ to Eq. (3.4) in a class of functions

$$w(\mathbf{x}) = \xi_A(\mathbf{x}) w_A, \qquad \mathbf{x} \in \Omega, \tag{4.1}$$

(subscripts A, B run over $0, 1, ..., N$, summation convention holds), where $\xi_A(\cdot)$ are linearly independent, sufficiently regular functions. Let us assume that the plate material is isotropic. Hence

$$G_{\alpha\beta\gamma\delta} = B\, I_{\alpha\beta\gamma\delta}, \qquad B = \frac{2Eh^3}{3(1-\nu^2)},$$

$$I_{\alpha\beta\gamma\delta} = \frac{1-\nu}{2}\left(\delta_{\alpha\gamma}\,\delta_{\beta\delta} + \delta_{\alpha\delta}\,\delta_{\beta\gamma}\right) + \nu\,\delta_{\alpha\beta}\,\delta_{\gamma\delta},$$

where E, ν stand for Young's modulus and Poisson's ratio, respectively. Applying constraints (4.1) to (3.4) and denoting

$$\begin{aligned} &F_{AB} \equiv \int_\Omega \xi_A\,\xi_B\,dA, \qquad E_{AB} \equiv \oint_{\partial\Omega} \xi_A\,\xi_B\,dS, \qquad C_{AB} \equiv \int_\Omega \xi_{A,\alpha}\,\xi_{B,\alpha}\,dA, \\ &I_{AB} \equiv I_{\alpha\beta\gamma\delta} \int_\Omega \xi_{A,\alpha\beta}\,\xi_{B,\gamma\delta}\,dA, \qquad \mu \equiv \frac{m}{s}\frac{1}{L}, \qquad \beta \equiv \frac{B}{s}\frac{1}{L^4}, \end{aligned} \tag{4.2}$$

we obtain the discretized model equations

$$(F_{AB} + \mu L E_{AB} + \mu^2 L^2 C_{AB} + \beta L^4 I_{AB}) w_B = \frac{1}{s} \int\limits_{\Omega} p \xi_A \, dA. \tag{4.3}$$

At the same time, the total strain energy function (3.6) is given by

$$\frac{1}{2}(F_{AB} + \mu L E_{AB} + \mu^2 L^2 C_{AB} + \beta L^4 I_{AB}) w_A w_B \, . \tag{4.4}$$

It can be seen that solutions to the governing equations (4.3) depend on non-dimensional parameters μ, β which are restricted by positive definiteness conditions of the quadratic form (4.4). In the framework of the asymptotic approximation of the above 2D-model, the terms involving μ drop out from the aforementioned equation.

5. Example

Let us assume $\Omega = (-L, L) \times (-L, L)$ and $p = \text{const}$. Using the discretized model, we assume that

$$\xi_0(\mathbf{x}) = 1, \qquad \xi_1(\mathbf{x}) = \frac{1}{2}(x_1)^2, \qquad \xi_2(\mathbf{x}) = \frac{1}{2}(x_2)^2$$

and

$$w(\mathbf{x}) = w_0 + \frac{1}{2}\left[(x_1)^2 + (x_2)^2\right] w_1 \, . \tag{5.1}$$

Hence we deal with a uniformly loaded square plate where $w_1 = w_2$ and Eqs. (3.5) yield

$$\begin{aligned}
R &= w_0 + \frac{1}{2}\left[(x_1)^2 + (x_2)^2\right] w_1 - 2\mu^2 L^2 w_1 - \frac{p}{s} \quad \text{in } \Omega, \\
M_\alpha &= -\beta(1+\nu) L^4 w_1 n_\alpha \quad \text{on } \partial\Omega, \\
Q &= \mu L \left\{ w_0 + \frac{1}{2}\left[(x_1)^2 + (x_2)^2\right] w_1 \right\} + \mu^2 L^2 \left[x_1 n_1(\mathbf{x}) + x_2 n_2(\mathbf{x})\right] w_1 \quad \text{on } \partial\Omega.
\end{aligned}$$

In the framework of the asymptotic model (i.e. neglecting the terms involving μ in Eqs. (4.3)), we obtain solutions to the above equations in the trivial form $w_0 = p/s$, $w_1 = 0$. In this case $R = 0$, $M_\alpha = 0$, $Q = 0$ (hence also $Q - M_{\zeta,\zeta} = 0$, $M_\eta = 0$) and in the framework of the asymptotic approximation, the above trivial solution is exact. Thus we conclude that the deflection $w(\mathbf{x})$ of the plate given above can constitute a good approximation also for small values of μ provided that $L^2 w_1 / w_0 < \varepsilon$, where ε is a small positive number, i.e. $1+\varepsilon \cong 1$. The above inequality can be treated as an applicability criterion of a simple discretized model introduced in this example. The governing equations (4.3) have now the form

$$\begin{aligned} &3(1+2\mu)\frac{w_0}{L} + (1+4\mu)Lw_1 = \frac{3p}{sL}, \\ &(1+4\mu)\frac{w_0}{L} + \left(\frac{7}{15} + \frac{14}{5}\mu + 2\mu^2 + 6\beta\right) Lw_1 = \frac{p}{sL}, \end{aligned} \tag{5.2}$$

which under notation

$$N \equiv 2(1+8\mu+46\mu^2+30\mu^3) + 90(1+2\mu)\beta$$

yield the simple formulae for deflections w_0 and curvatures $w_1 = w_2$ for the plate under consideration

$$\begin{aligned} \frac{w_0}{L} &= \frac{1}{1+2\mu}\left(1 + \frac{10\mu(1+4\mu)}{N}\right)\frac{p}{sL}, \\ Lw_1 &= -\frac{30\mu}{N}\frac{p}{sL}. \end{aligned} \tag{5.3}$$

The above simple formulas make it possible to evaluate the effect of non-dimensional moduli μ and β on the plate deformation. In order to evaluate the scope of applicability of (5.3), let us introduce non-dimensional residuals

$$R(0)\frac{s}{p}, \qquad M_\alpha \frac{s}{p\beta L^2}$$

and assume that their absolute values do not exceed a certain value $\varepsilon > 0$, which is small as compared to 1. Using the notations

$$\begin{aligned} \overline{w}_0 &\equiv \frac{w_0 s}{p} = \frac{1}{1+2\mu}\left(1 + \frac{10\mu(1+4\mu)}{N}\right), \\ \overline{w}_1 &\equiv -\frac{w_1 L^2 s}{p} = -\frac{30\mu}{N}, \end{aligned}$$

we obtain

$$\left|\overline{w}_0 - 2\mu^2\overline{w}_1 - 1\right| < \varepsilon, \qquad |\overline{w}_1| < \varepsilon. \tag{5.4}$$

The above conditions have to be treated as restrictions imposed on parameters μ and β for which formulae (5.3) have a physical meaning, i.e. plate deflections can be postulated in the form (5.1). For $\mu < 1$ conditions (5.4) yield

$$\frac{15\mu}{1+45\beta} + \mathcal{O}(\mu^2) \le \varepsilon.$$

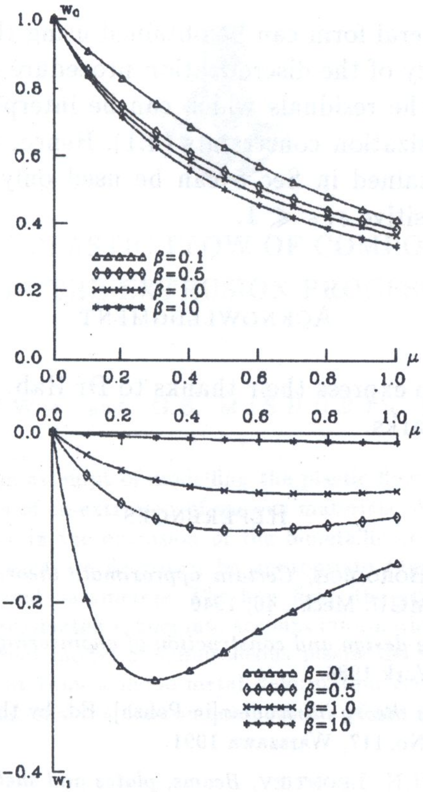

FIG. 3.

Neglecting terms $\mathcal{O}(\mu)$ for sufficiently small μ and setting $\varepsilon = \mu$ we arrive at $\beta > 0.31$ which is a physical reliability condition for the obtained solution. The diagrams of non-dimensional quantities $\overline{w}_0$, $\overline{w}_1$ are shown in Fig. 3 for different values of μ and β. Hence we conclude that the diagrams in Fig. 3 for $\beta = 0.1$ and small μ have only a formal meaning since they do not satisfy the aforementioned condition.

6. Conclusions

The proposed simplified 2D-model of thin plates resting on an elastic layer can be applied in problems in which the approximations introduced at the beginning of Sec. 3 are reliable. Such situation takes place if deflections along the plate boundary are not oscillating. The main feature of the proposed 2D-model is that the obtained variational equation after discretization leads to a system of linear algebraic equations with a symmetric matrix of coefficients. This system presented here in a general form can be obtained using the known finite element procedure. The accuracy of the discretization procedure can be evaluated *a posteriori* by calculating the residuals which can be interpreted as reaction forces maintaining the discretization constraints (4.1). Hence, the simple formulae for plate deformations obtained in Sec. 5 can be used only if conditions (5.4) are satisfied for a small positive ε, $\varepsilon \ll 1$.

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