ENGINEERING TRANSACTIONS • *Engng. Trans.* • **45**, 2, 237–250, 1997
Polish Academy of Sciences • Institute of Fundamental Technological Research

FREE VIBRATIONS OF A TAPERED CANTILEVER BEAM WITH ECCENTRICALLY CONCENTRATED MASSES AND INTERMEDIATE SUPPORTS

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The dynamic behaviour of beams with linearly varying cross-section, concentrated masses and intermediate supports has been investigated by means of two exact approaches. In the first case, a general purpose variational method (the so-called Whole Element Method) has been employed, whereas the second approach is classically expressed in terms of Bessel functions. The agreement between these different methods of analysis is illustrated by means of numerical examples.

1. Introduction

The free vibration frequencies of Euler-Bernoulli beams carrying concentrated masses have been considered by various authors, following different analytical and/or numerical approaches. The beam with constant cross-section has been extensively examined in the presence of masses at arbitrary positions and non-classical boundary conditions. Approximate solutions and effective numerical procedures were presented by Jacquot and Gibson [1], Parnell and Cobble [2]. A simplified approach has been proposed by Gürgoze [3–5], where a powerful extension of the Dunkerley and Southwell [9] methods is shown to be well adapted to the problem. An alternative procedure in terms of Green functions has been proposed by Xu and Chen [8].

Exact solutions are also available for cantilever tapered beams with a tip mass, starting from the classical analyses of Mabie and Rogers [7], where the particular tapered ratio allows a solution in terms of Bessel functions. Other solutions by Yang [13], Craver and Jampala [17] and Auciello [14] should also be noted, in which the influence of intermediate supports is taken into account.

A number of approximate variational solutions should also be mentioned, as, for example, the very accurate result by Grossi and Bhat [19] and Auciello [16], who used a Rayleigh - Ritz method in terms of orthogonal polynomials.

The aim of the present paper is to examine the dynamic behaviour of beams carrying concentrated masses in the presence of intermediate supports. A general variational method is applied, the so-called Whole Element Method (WEM), [10 – 12], which can be adopted to solve a wide range of linear and nonlinear boundary value problems. The particular system of the paper has been chosen in order to justify the name of the method, and even because it is possible to compare the results with an exact data in terms of Bessel functions.

2. A VARIATIONAL APPROACH TO THE PROBLEM

A propped cantilever beam with intermediate support and eccentric tip mass has been investigated. Basically, the method is attempted to minimize an *ad hoc* functional by adopting trial function $v(x)$ which is continuous with continuous first derivative in the definition domain $0 \le x \le L$.

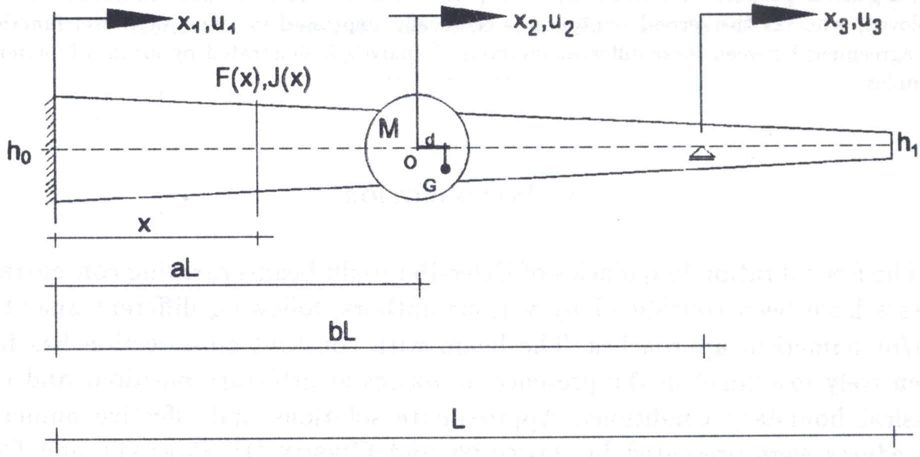

FIG. 1. Geometry of the system considered.

Let us consider a beam with linearly varying cross-section, whose geometrical data can be deduced from Fig. 1. Let E be the Young modulus, ϱ the mass density, J_0 and F_0 the second moment of area and the cross-section area of the beam at the abscissa $x = 0$, respectively. At the generic abscissa x, it will be:

$$(2.1)\qquad \begin{aligned} J(x) &= \left[1 + \varepsilon\left(\frac{x}{L} - 1\right)\right]^3 J_0 = h(x)J_0, \\ F(x) &= \left[1 + \varepsilon\left(\frac{x}{L} - 1\right)\right] F_0 = f(x)F_0, \end{aligned}$$

where $\varepsilon = 1 - h_0/h_1$. The presence of the applied mass is taken into account by means of the following nondimensional parameters:

$$(2.2)\qquad \begin{aligned} R &= \frac{M}{\varrho F_0 L}, & \mu &= \frac{M}{M_v}, & & \\ r &= \frac{i_{\varrho OM}}{L}, & D &= \frac{d}{L}, & k^2 &= D^2 + i_{\varrho OM}^2, \end{aligned}$$

where M_v is the total mass of the beam, $i_{\varrho OM}$ is the radius of gyration and d is the eccentricity with respect to the point O where the mass is attached.

If $v = v(x)$ are the transverse displacements of the beam, and ω the circular frequency, the functional will be written as:

$$(2.3)\qquad \Pi[v] = \Pi_1[v] - \Omega^2 \Pi_2[v],$$

where

$$(2.4)\qquad \Pi_1[v] \equiv \Pi_{11}[v] - \Omega^2 \Pi_{12}[v] = \left\| \sqrt{h}\, v'' \right\|^2 - \Omega^2 \left\| \sqrt{f}\, v \right\|^2,$$

$$(2.5)\qquad \Pi_2[v] \equiv R\left[v'^2(a) r^2 + v^2(a) + 2v'(a)v(a)D \right],$$

and

$$(2.6)\qquad \Omega^2 = \omega^2 \frac{\varrho F_0 L^4}{E I_0}.$$

In these equations the primes denote differentiation with respect to the spatial variable x. Moreover, we shall put:

$$(2.7)\qquad \begin{aligned} (P,Q) &\equiv \int_0^1 P(\xi) Q(\xi)\, d\xi, \\ \|P\|^2 &\equiv (P,P) = \int_0^1 P^2(\xi)\, d\xi < \infty, \end{aligned}$$

where $\xi = x/L$ and $P = P(x)$, $Q = Q(x)$ are two square integrable functions in $0 \le x \le L$.

2.1. The trial functions

We shall assume as approximate solution the following function:

$$(2.8)\qquad v_M = v_M(x) = -\sum_{i=1}^{M} \frac{A_i}{\beta_i^2} \sin \beta_i x + A_0 x + B, \qquad \beta_i = i\pi.$$

As can be seen, it is an extended Fourier series, such that if $v(x)$ and $v'(x)$ are continuous functions, in $0 \leq x \leq 1$ and $v''(x)$ is a square integrable function, then:

$$|v_M - v| \to 0, \qquad M \to \infty, \quad \forall x, \tag{2.9}$$

$$|v'_M - v'| \to 0, \qquad M \to \infty, \quad \forall x, \tag{2.10}$$

$$|v''_M - v''| \to 0, \qquad M \to \infty, \quad \forall x. \tag{2.11}$$

In other words, uniform convergence of v and v' and convergence in the mean of v'' must be assured, in the domain of interest. As usual,

$$A_i = 2\int_0^1 v(x)\sin\beta_i x\, dx, \qquad A_0 = v(1) - v(0), \qquad B = v(0). \tag{2.12}$$

A sequence which satisfies the above mentioned conditions will be called an extreming sequence.

Following the outlined steps, we obtain $v(x)$ and its derivative as functions of the unknown parameter Ω. This parameter can in turn be determined by imposing the stationarity of the functional $F[v_M]$, thus allowing the determination of the values A_i, A_0 and B.

Of course, the sequence v_M must satisfy the boundary conditions, which in this particular case read:

$$v(0) = 0, \qquad v'(0) = 0, \qquad v(b) = 0. \tag{2.13}$$

From these conditions we can immediately deduce $B = 0$ and

$$A_0 - \sum_{i=1}^{M} \frac{A_i}{\beta_i} = 0, \qquad A_0 b - \sum_{i=1}^{M} \frac{A_i}{\beta_i^2}\sin\beta_i b = 0. \tag{2.14}$$

It is worth noting that the W.E.M. cannot be considered as a particular Rayleigh - Ritz method, because the geometric boundary conditions must be satisfied only by the final sequence, but not necessarily by its single components.

According to Eq. (2.3) it is possible to write:

$$\Pi[v_M] = \Pi_1[v_M] - \Omega^2\Pi_2[v_M] + \Pi_3[v_M] \tag{2.15}$$

whereas

$$\Pi_3[v_M] = -2\lambda\left(A_0 - \sum_{i=1}^{M}\frac{A_i}{\beta_i}\right) - 2\mu\left(A_0 b - \sum_{i=1}^{M}\frac{A_i}{\beta_i^2}\sin\beta_i b\right) \tag{2.16}$$

with λ and μ being Lagrange multipliers.

3. Theorems and corollaries

3.1. Theorems

Generally speaking, an eigenvalue problem for a linear differential equation leads to the following expression:

$$\Pi[v] = 0 \tag{3.1}$$

and

$$\Omega^2 = \frac{\Pi_1[v] + \Pi_{11}[v]}{\Pi_2[v] + \Pi_{12}[v]} . \tag{3.2}$$

If $v = v(x)$ is the solution of the problem, then Ω is the exact eigenvalue. As a simple extension to the previous formula, it is possible to define the so-called Rayleigh quotient:

$$\Omega^2_M = \frac{\Pi_1[v_M] + \Pi_{11}[v_M] + \Pi_3[v_M]}{\Pi_2[v_M] + \Pi_{12}[v_M]} , \tag{3.3}$$

for which it is possible to prove several interesting properties. For example, the following theorems hold:

THEOREM 1. *The eigenvalue Ω^2 assumes an extremal value among the eigenvalues Ω^2_M if v_M is an extreming sequence, i.e. $\Omega^2_M = \Omega^2$, $M \to 0$.*

The proof can be read in several textbooks [10] and it is not given here for the sake of brevity.

COROLLARY 1. If Ω^2 is an extremal value among the eigenvalues Ω^2, also Ω^2 will be an extremal value with an *ad hoc* choice of the constant.

The following theorem is also true [11–12].

THEOREM 2. *If the eigenvalue Ω^2 assumes an extremal value among the eigenvalues Ω^2_M where v_M is a sequence not necessarily extreming but satisfying the essential boundary conditions (i.e. involving only v and v'), then $v(x)$ must be the classical solution that satisfies the differential equation.*

It is worth noting that this result yields the exact solution of the problem, i.e. the frequencies and their corresponding modal shapes agree with those obtained in the classical solution.

3.2. Applications

According to Theorem 1, the extremal condition for Ω^2_M can be written as

$$\delta\Omega^2_M = 0, \tag{3.4}$$

where δ denotes the variation with respect to the series coefficients. Equation (3.4) is equivalent to the following condition:

$$\delta\Pi[v_M] = 0 \tag{3.5}$$

or, more explicitly:

$$\begin{aligned}(hv'_M, \delta v''_M) &- \Omega^2(f_M v, \delta v_M)\\ &+ R\left[r^2 v'_M(a)\delta v_M(a) + v(a)\delta v_M(a) + D(v_M(a)\delta v_M(a) + v_M(a)\delta v'_M(a))\right]\\ &- \lambda\left(\delta A_0 - \sum_{i=1}^{M}\frac{\delta A_i}{\beta_i}\right) - \mu\left(b\delta A_0 - \sum_{i=1}^{M}\frac{\delta A_i}{\beta_i^2}\sin\beta_i b\right)\\ &- \delta\lambda\left(A_0 - \sum_{i=1}^{M}\frac{A_i}{\beta_i}\right) - \delta\mu\left(A_0 b - \sum_{i=1}^{M}\frac{A_i}{\beta_i^2}\sin\beta_i b\right) = 0.\end{aligned} \tag{3.6}$$

After some algebra it is possible to write

$$\begin{aligned}-\Omega^2&\left\{\theta A_0 - \sum_j A_j\gamma_j + R\left[r^2\left(A_0 - \sum_j\frac{A_j}{\beta_j}\cos\beta_j a\right)\right.\right.\\ &+\left(a^2 A_0 - a\sum_j\frac{A_j\sin\beta_j a}{\beta_j^2}\right) + D\left(\left(aA_0 - a\sum_j\frac{A_j}{\beta_j}\cos\beta_j a\right)\right.\\ &\left.\left.\left.+\left(a\sum_j\frac{A_j}{\beta_j}\cos\beta_j a\right) + \left(aA_0 - \sum_j\frac{A_j}{\beta_j^2}\sin\beta_j a\right)\right)\right]\right\} - \lambda - \mu b = 0,\end{aligned} \tag{3.7}$$

where we have denoted:

$$\theta = (f(x)x, x), \qquad \gamma_j = \frac{f(x)x_j\sin\beta_j x}{\beta_j^2}.$$

Moreover, the following quantities can be introduced

$$\sigma_{ij} = (h(x)\sin\beta_j x, \sin\beta_i x), \qquad \tau_{ij} = \frac{(f(x)\sin\beta_j x, \sin\beta_i x)}{\beta_i^2\beta_j^2},$$

$$\mu_{ij} = \frac{\sin\beta_i a\sin\beta_j a}{\beta_i^2\beta_j^2}, \qquad k_{ij} = \frac{\cos\beta_i a\cos\beta_j a}{\beta_i\beta_j}, \qquad \chi_{ij} = \frac{\cos\beta_j a\sin\beta_i a}{\beta_i^2\beta_j},$$

so that Eq. (3.7) becomes:

$$(3.8)\qquad \sum_j A_j\sigma_{ij} - \Omega^2\Bigg\{R\Bigg[r^2\Bigg(-A_0\frac{\cos\beta_i a}{\beta_i}+\sum_j A_j k_{ij}\Bigg) + \Bigg(-aA_0\frac{\sin\beta_i a}{\beta_i^2}+\sum_j A_j\mu_{ij}\Bigg) + D\Bigg(-A_0\frac{\sin\beta_i a}{\beta_i^2}+\sum_j A_j\chi_{ij} - aA_0\frac{\cos\beta_i a}{\beta_i}+\sum_j A_j\chi_{ij}\Bigg)\Bigg] + \sum_j A_j\tau_{ij} - A_0\gamma_i\Bigg\} + \frac{\lambda}{\beta_i} + \mu\frac{\sin\beta_i b}{\beta_i^2} = 0.$$

Equations (3.8) and (2.14) can be used in an iterative way, in order to find the frequency parameters and the corresponding vibration modes.

4. The classical approach to the problem

The equation of motion of the considered example can be written as

$$(4.1)\qquad \frac{d^2}{dx_i^2}\left[EI(x_i)\frac{d^2w_i}{dx_i^2}\right] - \varrho F(x_i)\omega^2 w_i = 0, \qquad i = 1,2,3,$$

where we have adopted the reference axes as in Fig. 1. Moreover,

$$0 \le x_i \le aL, \qquad 0 \le x_2 \le (b-a)L, \qquad 0 \le x_3 \le (1-b)L.$$

In order to simplify the procedure, it is convenient to adopt the following non-dimensional quantities:

$$(4.2)\quad u_1 = 1+\varepsilon\left(\frac{x_1}{L}-1\right), \qquad u_2 = 1+\varepsilon\left(a+\frac{x_2}{L}-1\right), \qquad u_3 = 1+\varepsilon\left(b+\frac{x_3}{L}-1\right),$$

where $\varepsilon = 1 - h_0/h_1$, and consequently the area and moment of inertia of the cross-section can be expressed as functions of the quantities at the free end as follows:

$$(4.3)\qquad F(x_i) = F_1 u_i\,, \qquad J(x_i) = J_1 u_i^3\,.$$

Inserting Eqs. (4.2)–(4.3) into Eq. (4.1) and simplifying it is possible to write:

$$(4.4)\qquad u_i^2 v_i^{IV} + 6u_i v_i^{III} + 6v_i^{II} - p_a^4 v_i = 0, \qquad i = 1,2,3,$$

where $(I) = d/du_i$ and

$$(4.5)\qquad p_a = -\frac{p}{\varepsilon}\,, \qquad p = \sqrt[4]{\omega^2\frac{\varrho F_1 L^4}{EJ_1}}\,.$$

The solutions of Eq. (4.4) are well-known, [18], and can be conveniently expressed in terms of Bessel functions. It is:

$$(4.6)\qquad w_i(u_i) = u_i^{-0.5}\Big[C_{1i}J_1(2p_a u_i^{0.5}) + C_{2i}Y_1(2p_a u_i^{0.5}) + C_{3i}I_1(2p_a u_i^{0.5}) + C_{4i}K_1(2p_a u_i^{0.5})\Big],$$

where J, Y, I, K are the Bessel functions of order 1, C_{ij} are 12 unknown constants to be determined by imposing the boundary conditions at the clamped end, at the mass, at the support, and at the free end. Therefore, it is:

At $x_1 = 0 \rightarrow u_1 = 1 - \varepsilon$,

$$v_1 = 0, \qquad v_1^I = 0.$$

At $x_2 = 0$ and $x_1 = aL \rightarrow u_1 = u_2 = 1 + \varepsilon(a-1)$,

$$v_1 = v_2 \qquad v_1^I = v_2^I,$$

$$v_1^{III} + 3u_1^{-1}v_1^{II} + \mu p^4 \frac{2-\varepsilon}{2\varepsilon^3} u_1^{-3} v_1 - 3u_2^{-1} v_2^{II} - v_2^{III} = 0,$$

$$v_1^{II} - v_2^{II} - k^2 p^4 u_1^{-3} \mu \frac{2-\varepsilon}{2\varepsilon} v_1^I = 0.$$

At the support $x_2 = (b-a)L$, $x_3 = 0 \rightarrow u_2 = u_3 = 1 + \varepsilon(b-1)$,

$$v_2 = v_3\,, \qquad v_2^I = v_3^I, \qquad v_2^{II} = v_3^{II}, \qquad v_2 = 0.$$

Finally, at the free end $x_3 = (1-b)L \rightarrow u_3 = 1$,

$$v_3^{II} = 0, \qquad v_3^{III} + 3v_3^{II} = 0.$$

An homogeneous system of linear equations is obtained by substituting Eqs. (4.6) and their derivatives into these conditions. A nontrivial solution is obtained by assuming

$$(4.7)\qquad \det \mathbf{A} = 0$$

whose non-zero terms are given in the Appendix.

Equation (4.7) is the frequency equation, and can be numerically solved with respect to the nondimensional parameter p by applying the usual procedure.

5. Numerical results

As already mentioned in Sec. 3, it is possible to find frequencies and vibrational modes in an iterative way, by using Eqs. (3.8) and (2.14). The coefficients

A_0, A_i, λ and μ can be calculated for each value of Ω. This procedure has been applied to the beam in Fig. 1, by using a series as in Eq. (2.8) with 1000 terms. The results have been given in Table 1, and a comparison with the classical results are also given, as obtained from Eq. (4.7).

It is interesting to note that the W.E.M. results are always slightly greater than the exact results, in compliance with the variational nature of the procedure [20–21].

Table 1. Non-dimensional frequency for $\alpha = 2$, $k = 0$, $a = 0.4$.

		p_1		p_2	
b	μ	Exact	W.E.M.	Exact	W.E.M.
	0	6.04322	6.04401	8.65786	8.65890
0.8	0.2	5.60499	5.60606	8.33893	8.34055
	1	4.63604	4.63703	8.02837	8.02853
	0	4.95987	4.96064	8.66341	8.66499
1	0.2	4.70599	4.70690	8.10953	8.11060
	1	4.04872	4.04919	7.46482	7.65245

Some noticeable discrepancies can be noted for the second free frequency, and for $b = 1$, $\mu = 1$. This is due to some numerical inaccuracies of the W.E.M.

For the sake of completeness, in Tab. 2 the exact nondimensional frequencies have been reported as functions of the parameter k.

Table 2. As Tab. 1 for $\mu = 1$, $a = 0.8$ and various k.

k	p_1	p_2	p_3	p_4
0.2	4.474557	8.260157	13.366113	16.140356
0.4	5.131388	8.212645	13.316203	16.109490
0.6	5.003794	8.205683	13.307044	16.103904
0.8	4.968777	8.203356	13.303844	16.101959
1	4.954117	8.202297	13.302365	16.101060

The influence of the a and k parameters on the first three nondimensional frequencies has been illustrated in Figs. 2, 3, 4 for a particular taper ratio $\alpha = 2$. It is interesting to observe the curves p_2 and p_3 as functions of k, where they practically coincide for $k > 0.4$. On the other hand, noticeable discrepancies can be observed for $0 < k < 0.2$.

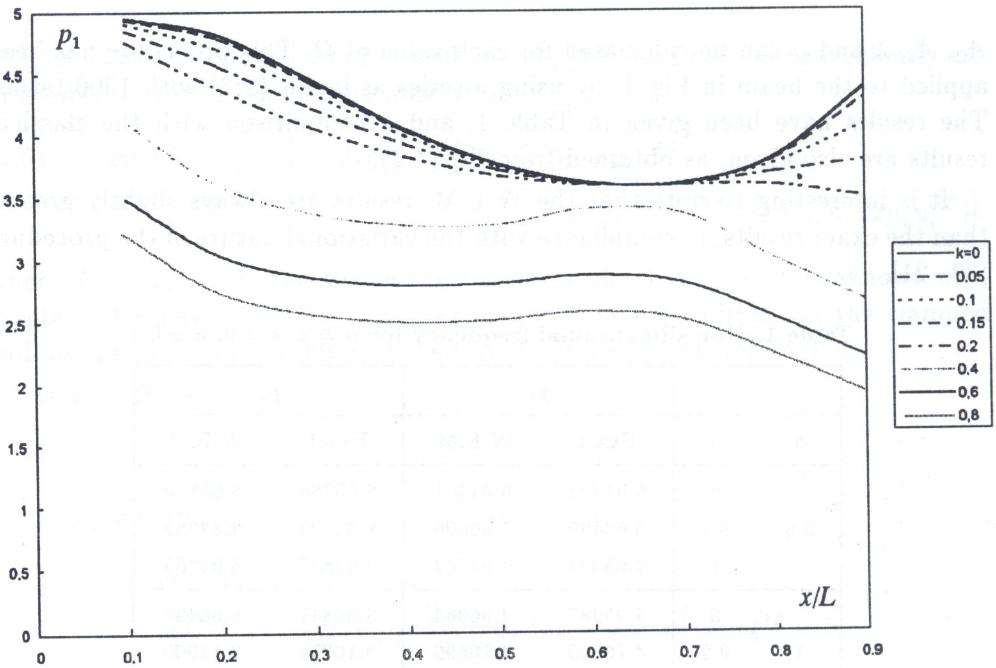

FIG. 2. First exact frequency coefficients for $\alpha = 2$, $b = 1$ and various k, a.

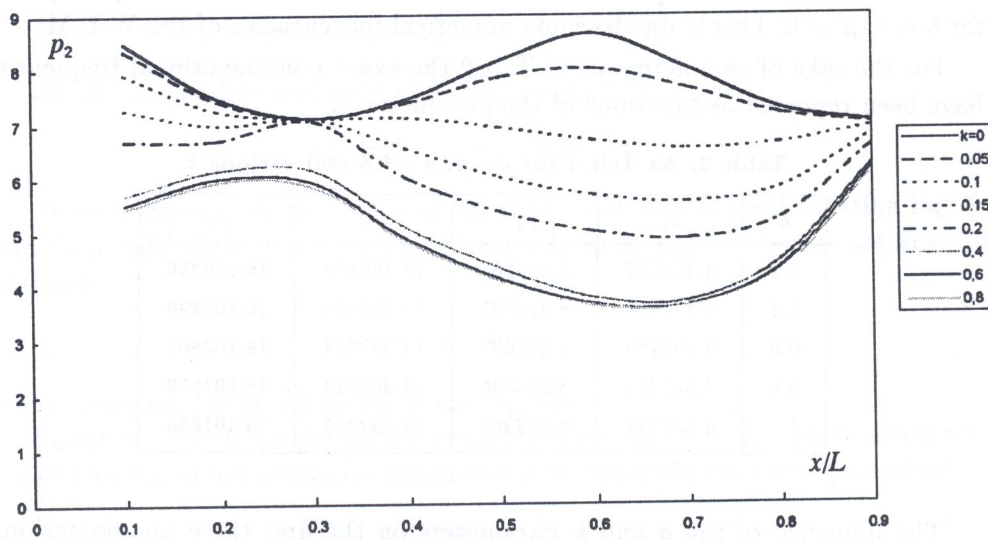

FIG. 3. As Fig. 2; second frequency coefficients.

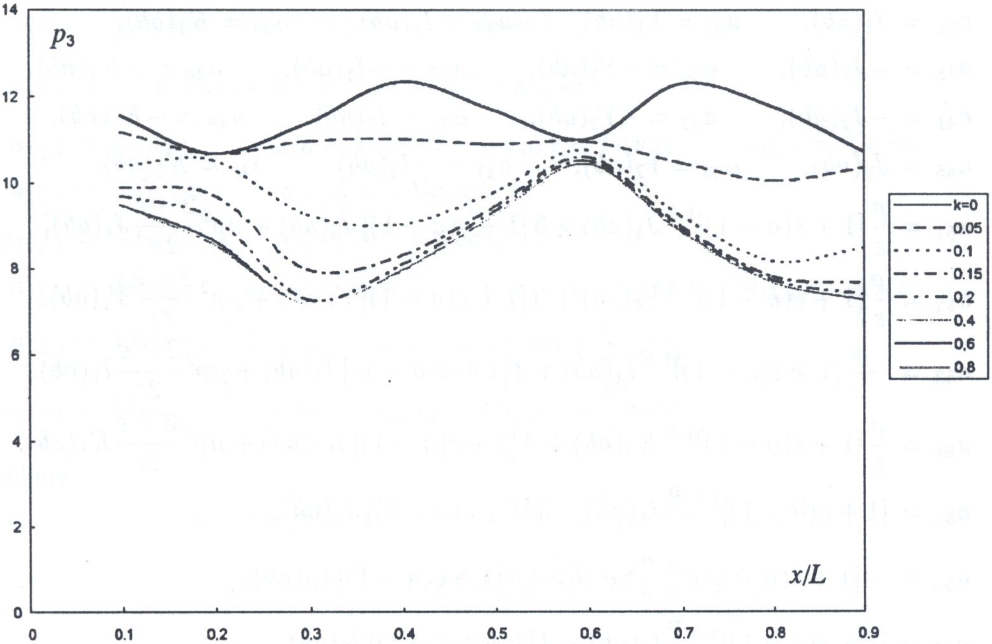

FIG. 4. As Fig. 2; third frequency coefficients.

6. CONCLUSIONS

The proposed general-purpose variational method is particularly useful for the dynamic analysis of structures subjected to heavy masses. A single trial function can be used, so that its application to complex structures is almost immediate. Moreover, the essential boundary conditions must be satisfied only by the final sequence, but not by its individual terms, as in the classical Rayleigh - Ritz approach.

In this paper no attempts have been made in order to deduce the vibrational modes, because the problem turns out to be not self-adjoint, and a cumbersome Gram - Schmidt-like procedure should be adopted in order to render the eigenvectors orthonormal. On the other hand, the emphasis of the paper has been placed on the adopted variational procedure.

APPENDIX

$$a_{11} = J_1(aa), \qquad a_{12} = Y_1(aa), \qquad a_{13} = I_1(aa), \qquad a_{14} = K_1(aa),$$
$$a_{21} = -J_2(aa), \qquad a_{22} = -Y_2(aa), \qquad a_{23} = I_2(aa), \qquad a_{24} = -K_2(aa),$$

$$a_{31} = J_1(ab), \qquad a_{32} = Y_1(ab), \qquad a_{33} = I_1(ab), \qquad a_{34} = K_1(ab),$$
$$a_{35} = -J_1(ab), \qquad a_{36} = -Y_1(ab), \qquad a_{37} = -I_1(ab), \qquad a_{38} = -K_1(ab),$$
$$a_{41} = -J_2(ab), \qquad a_{42} = -Y_2(ab), \qquad a_{43} = I_2(ab), \qquad a_{44} = -K_2(ab),$$
$$a_{45} = J_2(ab), \qquad a_{46} = Y_2(ab), \qquad a_{47} = -I_2(ab), \qquad a_{48} = K_2(ab),$$
$$a_{51} = \frac{p}{\varepsilon}\left[1+\varepsilon(a-1)\right]^{1.5} J_4(ab) + 3\left[1+\varepsilon(a-1)\right] J_3(ab) + \mu p^2 \frac{2-\varepsilon}{2\varepsilon} J_1(ab),$$
$$a_{52} = \frac{p}{\varepsilon}\left[1+\varepsilon(a-1)\right]^{1.5} Y_4(ab) + 3\left[1+\varepsilon(a-1)\right] Y_3(ab) + \mu p^2 \frac{2-\varepsilon}{2\varepsilon} Y_1(ab),$$
$$a_{53} = -\frac{p}{\varepsilon}\left[1+\varepsilon(a-1)\right]^{1.5} I_4(ab) + 3\left[1+\varepsilon(a-1)\right] I_3(ab) + \mu p^2 \frac{2-\varepsilon}{2\varepsilon} I_1(ab),$$
$$a_{54} = \frac{p}{\varepsilon}\left[1+\varepsilon(a-1)\right]^{1.5} K_4(ab) + 3\left[1+\varepsilon(a-1)\right] K_3(ab) + \mu p^2 \frac{2-\varepsilon}{2\varepsilon} K_1(ab),$$
$$a_{55} = \left[1+\varepsilon(a-1)\right]^{1.5} \frac{p}{\varepsilon} J_4(ab) - 3\left[1+\varepsilon(a-1)\right] J_3(ab),$$
$$a_{56} = -\left[1+\varepsilon(a-1)\right]^{1.5} \frac{p}{\varepsilon} Y_4(ab) - 3\left[1+\varepsilon(a-1)\right] Y_3(ab),$$
$$a_{57} = \left[1+\varepsilon(a-1)\right]^{1.5} \frac{p}{\varepsilon} I_4(ab) - 3\left[1+\varepsilon(a-1)\right] I_3(ab),$$
$$a_{58} = -\left[1+\varepsilon(a-1)\right]^{1.5} \frac{p}{\varepsilon} K_4(ab) - 3\left[1+\varepsilon(a-1)\right] K_3(ab),$$
$$a_{61} = J_3(ab) - k^2\mu\left[1+\varepsilon(a-1)\right]^{-2.5} p^2(2-\varepsilon) J_2(ab),$$
$$a_{62} = Y_3(ab) - k^2\mu\left[1+\varepsilon(a-1)\right]^{-2.5} p^2(2-\varepsilon) Y_2(ab),$$
$$a_{63} = I_3(ab) + k^2\mu\left[1+\varepsilon(a-1)\right]^{-2.5} p^2(2-\varepsilon) I_2(ab),$$
$$a_{64} = K_3(ab) - k^2\mu\left[1+\varepsilon(a-1)\right]^{-2.5} p^2(2-\varepsilon) K_2(ab),$$
$$a_{65} = -J_3(ab), \qquad a_{66} = -Y_3(ab), \qquad a_{67} = -I_3(ab), \qquad a_{68} = -K_3(ab),$$
$$a_{75} = -J_2(ba), \qquad a_{76} = -Y_2(ba), \qquad a_{77} = I_2(ba), \qquad a_{78} = -K_2(ba),$$
$$a_{79} = J_2(ba), \qquad a_{7,10} = Y_2(ba), \qquad a_{7,11} = -I_2(ba), \qquad a_{7,12} = K_2(ba),$$
$$a_{85} = J_1(ba), \qquad a_{86} = Y_1(ba), \qquad a_{87} = I_1(ba), \qquad a_{88} = K_1(ba),$$
$$a_{89} = -J_1(ba), \qquad a_{8,10} = -Y_1(ba), \qquad a_{8,11} = -I_1(ba), \qquad a_{8,12} = K_1(ba),$$
$$a_{95} = \frac{1}{p^2\varepsilon}\left[1+\varepsilon(b-1)\right]^{-1.5} J_1(ba), \qquad a_{96} = \frac{1}{p^2\varepsilon}\left[1+\varepsilon(b-1)\right]^{-1.5} Y_1(ba),$$
$$a_{97} = \frac{1}{p^2\varepsilon}\left[1+\varepsilon(b-1)\right]^{-1.5} I_1(ba), \qquad a_{98} = \frac{1}{p^2\varepsilon}\left[1+\varepsilon(b-1)\right]^{-1.5} K_1(ba),$$
$$a_{99} = -\frac{p}{\varepsilon} J_4(ba) - 3\left[1+\varepsilon(b-1)\right]^{-0.5} J_3(ba),$$
$$a_{9,10} = -\frac{p}{\varepsilon} Y_4(ba) - 3\left[1+\varepsilon(b-1)\right]^{-0.5} Y_3(ba),$$

$$a_{9,11} = \frac{p}{\varepsilon} I_4(ba) - 3\left[1+\varepsilon(b-1)\right]^{-0.5} I_3(ba),$$

$$a_{9,12} = -\frac{p}{\varepsilon} K_4(ba) - 3\left[1+\varepsilon(b-1)\right]^{-0.5} K_3(ba),$$

$$a_{10,5} = J_3(ba), \qquad a_{10,6} = Y_3(ba), \qquad a_{10,7} = I_3(ba), \qquad a_{10,8} = K_3(ba),$$

$$a_{10,9} = -J_3(ba), \qquad a_{10,10} = -Y_3(ba),$$

$$a_{10,11} = -I_3(ba), \qquad a_{10,12} = -K_3(ba),$$

$$a_{11,9} = J_3(c), \qquad a_{11,10} = Y_4(c), \qquad a_{11,11} = -I_4(c), \qquad a_{11,12} = K_4(c),$$

$$a_{12,9} = 3J_3(c) + \frac{p}{\varepsilon} J_4(c), \qquad a_{12,10} = 3Y_3(c) + \frac{p}{\varepsilon} Y_4(c),$$

$$a_{12,11} = 3I_3(c) - \frac{p}{\varepsilon} I_4(c), \qquad a_{12,12} = 3K_3(c) + \frac{p}{\varepsilon} K_4(c),$$

where

$$aa = -2\frac{p}{\varepsilon}(1-\varepsilon)^{0.5}, \qquad ab = -2\frac{p}{\varepsilon}\left[1+\varepsilon(a-1)\right]^{0.5},$$

$$ba = -2\frac{p}{\varepsilon}\left[1+\varepsilon(b-1)\right]^{0.5}, \qquad c = -2\frac{p}{\varepsilon}.$$

ACKNOWLEDGMENT

The work of the first author was partially supported by Italian M.U.R.S.T. (60%).

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Received November 20, 1996; new version March 14, 1997.