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Research Paper

Estimation of the Torsional Rigidity of Orthotropic Solid Cross Section

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In this paper, two inequality relations are proven for the torsional rigidity of orthotropic elastic solid cross sections. By using the derived inequality relations, lower and upper bounds can be obtained for the torsional rigidity. All results of the paper follow from the Saint-Venant theory of uniform torsion. The presented bounding formulae are based on the mean value theorem of integral calculus.

Key words: Saint-Venant torsion; orthotropic; torsional rigidity; Barta-type inequality.

1. Introduction

This paper formulates two inequality relations for the torsional rigidity of elastic homogeneous orthotropic solid cross section. The presented inequality relations can be used to estimate the torsional rigidity, and they are different from the bounding formulas of torsional rigidity obtained by applying the minimum principles of the linear theory of elasticity [1–4]. They are based on the mean-value theorem of integral calculus,and these types of inequalities are called Barta's inequalities [5–8]. Barta's inequalities are very handsome tools for constructing rough bounds for the torsional rigidity and for eigenvalues. Although they have been used and refined for numerical applications [5, 6], they are mainly of theoretical interest. Barta extended his method of estimating torsional rigidity to multi-cell thin-walled cross sections by using the properties of the weighted arithmetic mean of a finite sum [9, 10]. In this paper, Barta-type inequalities will be derived for the torsional rigidity of orthotropic elastic solid cross sections. Four examples illustrate the application of the proven inequality relations.

2. Saint-Venant torsion of orthotropic solid cross section

Figure 1 shows the cross section of the orthotropic beam, which is denoted by A. The boundary curve of A is indicated by ∂A and the arc-length defined on ∂A is denoted by s, $\mathbf{t} = \mathbf{t}(s)$ and $\mathbf{n} = \mathbf{n}(s)$ represent the unit tangential and unit normal vectors of the boundary contour ∂A. Unit vectors in x and y directions are $\mathbf{e}_x$ and $\mathbf{e}_y$ and we have (Fig. 1)

$$\mathbf{t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = t_x\mathbf{e}_x + t_y\mathbf{e}_y = \frac{\mathrm{d}x}{\mathrm{d}s}\mathbf{e}_x + \frac{\mathrm{d}y}{\mathrm{d}s}\mathbf{e}_y, \tag{2.1}$$

$$\mathbf{n} = n_x\mathbf{e}_x + n_y\mathbf{e}_y = \frac{\mathrm{d}y}{\mathrm{d}s}\mathbf{e}_x - \frac{\mathrm{d}x}{\mathrm{d}s}\mathbf{e}_y. \tag{2.2}$$

Here, $\mathbf{r}(s) = x(s)\mathbf{e}_x + y(s)\mathbf{e}_y$ is the position vector of the boundary point P (Fig. 1). The shear moduli of orthotropic material in xz and yz directions are denoted by G_x and G_y. $U = U(x, y)$ is the Prandtl stress function of the considered orthotropic cross section. Here, we have $U = U(x, y)$ as the solution of the following Dirichlet-type boundary value problem [1–4]:

$$\frac{1}{G_y}\frac{\partial^2 U}{\partial x^2} + \frac{1}{G_x}\frac{\partial^2 U}{\partial y^2} = -2, \qquad (x, y) \in A, \tag{2.3}$$

$$U(x, y) = 0, \qquad (x, y) \in \partial A. \tag{2.4}$$

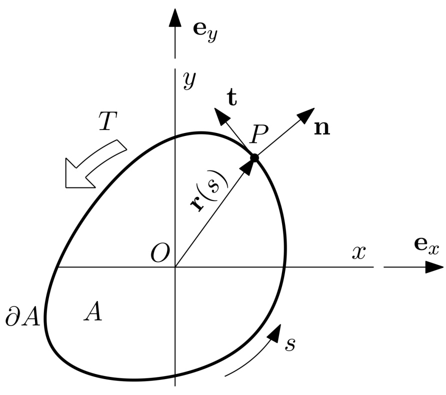

Fig. 1. Solid orthotropic cross section.

The shear stress vector $\boldsymbol{\tau}_z$ in terms of stress function $U = U(x, y)$ can be expressed as [1–4]

$$\boldsymbol{\tau}_z = \tau_{xz}\mathbf{e}_x + \tau_{yz}\mathbf{e}_y = \vartheta\left(\frac{\partial U}{\partial y}\mathbf{e}_x - \frac{\partial U}{\partial x}\mathbf{e}_y\right), \tag{2.5}$$

where ϑ is the rate of twist. The connection between the applied torque T and ϑ is as follows:

$$\vartheta = \frac{T}{S} \tag{2.6}$$

and we have the following formula for the torsional rigidity S:

$$S = 2\int_A U(x,y)\,\mathrm{d}A = \int_A \left[\frac{1}{G_y}\left(\frac{\partial U}{\partial x}\right)^2 + \frac{1}{G_x}\left(\frac{\partial U}{\partial y}\right)^2\right]\mathrm{d}A. \tag{2.7}$$

At first, we prove that

$$U(x,y) \geq 0, \qquad (x,y) \in \overline{A} = A \cup \partial A. \tag{2.8}$$

Let

$$X = x\sqrt[4]{\frac{G_y}{G_x}}, \qquad Y = y\sqrt[4]{\frac{G_x}{G_y}}, \qquad (x,y) \in \overline{A}. \tag{2.9}$$

Clearly, if $(x,y) \in A$, then

$$(X,Y) \in a = \left\{X,Y|X = x\sqrt[4]{\frac{G_y}{G_x}}, \quad Y = y\sqrt[4]{\frac{G_x}{G_y}}\right\}. \tag{2.10}$$

A simple computation based on Eqs (2.3), (2.4) and (2.9) gives

$$\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2} = -2\sqrt{G_x G_y}, \qquad (X,Y) \in a, \tag{2.11}$$

$$\begin{gathered} u(X,Y) = 0, \qquad (X,Y) \in \partial a, \\ (X,Y) \in \partial a = \left\{X,Y|X = x\sqrt[4]{\frac{G_y}{G_x}}, \quad Y = y\sqrt[4]{\frac{G_x}{G_y}}\right\}, \end{gathered} \tag{2.12}$$

in Eq. (2.12), $(x,y) \in \partial A$ and $(X,Y) \in \partial a$. The connection of the functions $U(x,y)$, $(x,y) \in \overline{A}$, and $u(X,Y)$, $(X,Y) \in \overline{a}$, is given by the following equation:

$$U(x,y) = U\left(X\sqrt[4]{\frac{G_x}{G_y}}, \quad Y\sqrt[4]{\frac{G_y}{G_x}}\right) = u\left(X,Y\right). \tag{2.13}$$

From Eqs (2.11)–(2.13) it follows that according to the properties of the Prandtl stress function of the homogeneous isotropic solid cross section we have

$$u(X,Y) \geq 0, \qquad (X,Y) \in a \cup \partial a. \tag{2.14}$$

Since $U(x,y) \geq 0$ in A and $U(x,y) = 0$ on ∂A, we have the derivative of the continuous function $U = U(x,y)$ in the direction of $\mathbf{n}$ is non-positive, that is,

$$\frac{\partial U}{\partial n} = \frac{\partial U}{\partial x} n_x + \frac{\partial U}{\partial y} n_y \leq 0, \qquad (x,y) \in \partial A. \tag{2.15}$$

Later on, we will use the following equations

$$\int_A \left[\frac{\partial}{\partial x}\left(\frac{1}{G_y} \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial y}\left(\frac{1}{G_x} \frac{\partial U}{\partial y} \right) \right] \mathrm{d}A = \int_{\partial A} \left(\frac{1}{G_y} \frac{\partial U}{\partial x} n_x + \frac{1}{G_x} \frac{\partial U}{\partial y} n_y \right) \mathrm{d}s = -2A, \tag{2.16}$$

$$\frac{n_x}{G_y} \frac{\partial U}{\partial x} + \frac{n_y}{G_x} \frac{\partial U}{\partial y} = \left(\frac{n_x^2}{G_y} + \frac{n_y^2}{G_x} \right) \frac{\partial U}{\partial n} \leq 0, \qquad (x,y) \in \partial A. \tag{2.17}$$

The validity of Eq. (2.17) follows from the system of Eqs (2.18) and (2.19):

$$\frac{\tau_{sz}}{\vartheta} = -\left(\frac{\partial U}{\partial x} n_x + \frac{\partial U}{\partial y} n_y \right) = -\frac{\partial U}{\partial n}, \tag{2.18}$$

$$\frac{\tau_{nz}}{\vartheta} = -\frac{\partial U}{\partial x} n_y + \frac{\partial U}{\partial y} n_x = 0. \tag{2.19}$$

The solution of the system of Eqs (2.18) and (2.19) for $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$ is as follows:

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial n} n_x, \qquad \frac{\partial U}{\partial y} = \frac{\partial U}{\partial n} n_y. \tag{2.20}$$

Substitution of Eqs $(2.20)_{1,2}$ into the left-hand side of Eq. (2.17) gives

$$\frac{n_x}{G_y} \frac{\partial U}{\partial x} + \frac{n_y}{G_x} \frac{\partial U}{\partial y} \leq 0, \qquad (x,y) \in \partial A. \tag{2.21}$$

3. Barta-type inequality for the torsional rigidity

We start with the following equation:

$$\left(\frac{1}{G_y} \frac{\partial^2 U}{\partial x^2} + \frac{1}{G_x} \frac{\partial^2 U}{\partial y^2} \right) V = -2V, \qquad (x,y) \in A, \tag{3.1}$$

where $V = V(x, y)$ can be differentiated at least twice in A and it is continuous in $\overline{A} = A \cup \partial A$, otherwise $V = V(x, y)$ is an arbitrary function of x and y. The left-hand side of Eq. (3.1) can be reformulated as:

$$(3.2)\quad V\frac{\partial}{\partial x}\left(\frac{1}{G_y}\frac{\partial U}{\partial x}\right) + V\frac{\partial}{\partial y}\left(\frac{1}{G_x}\frac{\partial U}{\partial y}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{V}{G_y}\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{V}{G_x}\frac{\partial U}{\partial y}\right) - \frac{1}{G_y}\frac{\partial V}{\partial x}\frac{\partial U}{\partial x} - \frac{1}{G_x}\frac{\partial V}{\partial y}\frac{\partial U}{\partial y}$$
$$= \frac{\partial}{\partial x}\left(\frac{V}{G_y}\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{V}{G_x}\frac{\partial U}{\partial y}\right)$$
$$- \frac{\partial}{\partial x}\left(\frac{U}{G_y}\frac{\partial V}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{U}{G_x}\frac{\partial V}{\partial y}\right) + \left(\frac{1}{G_y}\frac{\partial^2 V}{\partial x^2} + \frac{1}{G_x}\frac{\partial^2 V}{\partial y^2}\right)U.$$

Combining Eq. (3.1) with Eq. (3.2) gives

$$(3.3)\qquad 2\int_A V\,\mathrm{d}A = 2\int_A F(x, y)\,U\,\mathrm{d}A - \int_{\partial A}\left(\frac{n_x}{G_y}\frac{\partial U}{\partial x} + \frac{n_y}{G_x}\frac{\partial U}{\partial y}\right)V\,\mathrm{d}s,$$

where

$$(3.4)\qquad F(x, y) = -\frac{1}{2}\left(\frac{1}{G_y}\frac{\partial^2 V}{\partial x^2} + \frac{1}{G_x}\frac{\partial^2 V}{\partial y^2}\right).$$

Here, we have used

$$(3.5)\qquad U(x, y) = 0, \qquad (x, y) \in \partial A.$$

Since

$$(3.6)\quad U(x, y) \geq 0 \quad (x, y) \in A \quad \text{and} \quad -\left(\frac{n_x}{G_y}\frac{\partial U}{\partial x} + \frac{n_y}{G_x}\frac{\partial U}{\partial y}\right) \leq 0, \quad (x, y) \in \partial A,$$

from Eq. (3.3), it follows that

$$(3.7)\qquad 2\int_A V\,\mathrm{d}A \leq \overline{F}\,S + 2\,\overline{v}\,A,$$

$$(3.8)\qquad 2\int_A V\,\mathrm{d}A \geq \underline{F}\,S + 2\,\underline{v}\,A,$$

where

$$(3.9)\quad \overline{F} = \sup F(x, y),\ \ (x, y) \in A \cup \partial A, \quad \underline{F} = \inf F(x, y),\ \ (x, y) \in A \cup \partial A,$$

$$(3.10)\quad \overline{v} = \sup V(x, y),\ \ (x, y) \in \partial A, \qquad \underline{v} = \inf V(x, y),\ \ (x, y) \in \partial A.$$

Here, we note, if

$$\overline{F} = \underline{F} = F_0 \neq 0, \quad \text{and} \quad V(x,y) = 0, \quad (x,y) = 0, \tag{3.11}$$

then

$$S = \frac{2}{F_0} \int_A V \, \mathrm{d}A. \tag{3.12}$$

It is very easy to show that if

$$\overline{F} = F = 0 \quad \text{and} \quad V(x,y) = 0, \quad (x,y) \in \partial A, \tag{3.13}$$

then we have

$$V(x,y) \equiv 0, \quad (x,y) \in A \cup \partial A. \tag{3.14}$$

4. Examples

4.1. Cross section with straight line and hyperbola arc boundary curves

Figure 2 shows a cross section bordered by the straight line $x = b$, and a hyperbola arc defined by the formula

$$x^2 - p^2 y^2 - a^2 = 0. \tag{4.1}$$

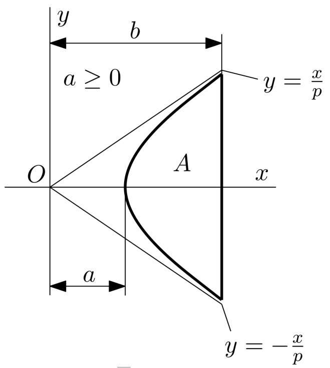

FIG. 2. Cross section bordered by a straight line and a hyperbola arc.

In Eq. (4.1), p is a parameter whose value determines the shape of the hyperbola arc. Let

$$V(x,y) = (x^2 - p^2 y^2 - a^2)(b - x), \quad b > a. \tag{4.2}$$

In this case, we have

$$F(x,y) = -\frac{1}{2}\left(\frac{1}{G_y}\frac{\partial^2 V}{\partial x^2} + \frac{1}{G_x}\frac{\partial^2 V}{\partial y^2}\right) = \left(\frac{3}{G_y} - \frac{p^2}{G_x}\right)x + b\left(\frac{p^2}{G_x} - \frac{1}{G_y}\right), \tag{4.3}$$

$$V(x,y) = 0, \quad (x,y) \in \partial A. \tag{4.4}$$

From Eq. (4.4), it follows that if

$$G_y = \frac{3}{p^2} G_x, \tag{4.5}$$

then we have

$$\overline{F} = \underline{F} = F_0 = \frac{2}{3}\frac{p^2}{G_x} b^2. \tag{4.6}$$

Application of formula (3.12) yields the following result for the torsional rigidity of the considered orthotropic cross section:

$$S = \frac{G_x}{10p^3 b}(R_1 - R_2), \tag{4.7}$$

$$R_1 = 10b^2\sqrt{(b^2-a^2)^3} + 15a^4 b \ln(b + \sqrt{b^2-a^2}), \tag{4.8}$$

$$R_2 = 8\sqrt{(b^2-a^2)^5} + 15ba^4\ln(a) + 15b^2a^2\sqrt{b^2-a^2}. \tag{4.9}$$

We note that the formula (4.7) for the case $a = 0$ gives the torsional rigidity of an isosceles triangle that is bordered by the two asymptotes of the hyperbola arc shown in Fig. 2 and by the straight line $x = b$. From formula (4.7), we obtain the following result for the torsional rigidity of the anisotropic isosceles orthotropic cross section:

$$S = \frac{1}{5} G_x \frac{b^4}{p} \quad \text{if} \quad G_y = \frac{3}{p^2} G_x. \tag{4.10}$$

4.2. Anisotropic isosceles right triangle cross section

Figure 3 illustrates the considered isosceles triangle cross section. The following numerical data are used:

$$\begin{aligned} &G_x = 8 \cdot 10^8 \text{ Pa}, \qquad && G_y = 25 \cdot 10^8 \text{ Pa}, \\ &a = 0.1 \text{ m}, && A = a^2 = 0.01 \text{ m}^2. \end{aligned} \tag{4.11}$$

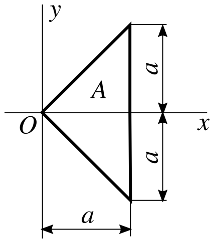

FIG. 3. Anisotropic isosceles triangle cross section.

Let

$$V(x,y) = y^2x - x^3 - ay^2 + ax^2. \tag{4.12}$$

This function has zero value on the boundary of the considered isosceles right triangle cross section. We have

$$\begin{gathered}\overline{F} = 8.5 \cdot 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{N}}, \qquad \underline{F} = 8 \cdot 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{N}},\\ 2\int_A V\,\mathrm{d}A = 1.3333333 \cdot 10^{-6}\ \mathrm{m}^5.\end{gathered} \tag{4.13}$$

Substituting the numerical results given by Eq. (4.12) into inequality relations (3.7) and (3.8), the following upper and lower bounds can be derived for the torsional rigidity:

$$S < 16666.666\ \mathrm{N}\cdot\mathrm{m}^2, \qquad S > 15686.27451\ \mathrm{N}\cdot\mathrm{m}^2. \tag{4.14}$$

4.3. Cross section with straight line and parabola arc boundary curves

Figure 4 shows an orthotropic cross section that is bordered by the straight line $y = 0$ and the parabola arc defined by the formula

$$\frac{y}{a} - \frac{x}{b}\left(1 - \frac{x}{b}\right) = 0, \quad 0 \le x \le b. \tag{4.15}$$

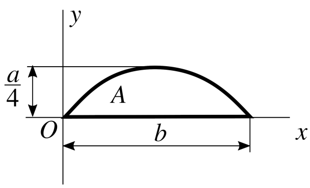

FIG. 4. A cross section bordered by a straight line and a parabola arc.

In this example, the following numerical data are used:

$$G_x = 0.9 \cdot 10^5 \ \mathrm{N/mm^2}, \qquad G_y = 0.7 \cdot 10^5 \ \mathrm{N/mm^2},$$
$$a = 6 \ \mathrm{mm}, \qquad b = 60 \ \mathrm{mm}.$$

Let

$$V = V(x,y) = y\left(\frac{y}{a} - \frac{x}{b}\left(1 - \frac{x}{b}\right)\right). \tag{4.16}$$

This function satisfies the following homogeneous boundary condition:

$$V(x,y) = 0, \quad (x,y) \in \partial A. \tag{4.17}$$

In this case, we have

$$\begin{gathered} \underline{F} = -1.8518519 \cdot 10^{-6} \ \frac{\mathrm{mm}}{\mathrm{N}}, \qquad \overline{F} = -1.857804 \cdot 10^{-6} \ \frac{\mathrm{mm}}{\mathrm{N}}, \\ 2\int_A V \, \mathrm{d}A = -5.142857143 \ \mathrm{mm}^3. \end{gathered} \tag{4.18}$$

By substituing the numerical results given by Eq. (4.17) into bounding formulae (3.7) and (3.8), we obtain the following upper and lower bounds for the torsional rigidity:

$$S < 2.7771429 \cdot 10^6 \ \mathrm{N \cdot mm^2}, \qquad S > 2.7682449 \cdot 10^6 \ \mathrm{N \cdot mm^2}. \tag{4.19}$$

We can get another lower bound for the torsional rigidity, which can be derived by applying the minimum principle of the complementary energy of elasticity [1–3]

$$S \geq 4\int_A U \, \mathrm{d}A - \int_A \left[\frac{1}{G_y}\left(\frac{\partial U}{\partial x}\right)^2 + \frac{1}{G_x}\left(\frac{\partial U}{\partial y}\right)^2\right] \mathrm{d}A, \tag{4.20}$$

where the statically admissible Prandtl stress function $U = U(x,y)$ satisfies the following boundary condition

$$U(x,y) = 0, \quad (x,y) \in \partial A. \tag{4.21}$$

Let

$$U = Cy\left(\frac{y}{a} - \frac{x}{b}\left(1 - \frac{x}{b}\right)\right). \tag{4.22}$$

A simple calculation shows that the inequality relation (4.20) gives the sharpest lower bound for the torsional rigidity if

$$C = -5.392296728 \cdot 10^5 \text{ N/mm}. \tag{4.23}$$

In this case, we have

$$S \geq 2.7731812 \cdot 10^6 \text{ N} \cdot \text{mm}^2. \tag{4.24}$$

This lower bound is smaller than the upper bound given by inequality $(4.19)_1$. This fact supports the validity and applicability of the proven bounding formulae (3.7) and (3.8).

5. Conclusions

Barta-type inequality relation was proven for the torsional rigidity of homogeneous orthotropic elastic beams with solid cross sections. The presented inequality can be used to estimate the torsional rigidity. Some properties of the Prandtl stress function for the solid orthotropic cross section were also proven. Examples illustrate the applications of the presented bounding formulae. In the first example, the exact value of the torsional rigidity was obtained.

Acknowledgments

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