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Research Paper

Investigation of Free Vibration and Buckling of Timoshenko Nano-beam Based on a General Form of Eringen Theory Using Conformable Fractional Derivative and Galerkin Method

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The purpose of this paper is to study the free vibration and buckling of a Timoshenko nano-beam using the general form of the Eringen theory generalized based on the fractional derivatives.

In this paper, using the conformable fractional derivative (CFD) definition the generalized form of the Eringen nonlocal theory (ENT) is used to consider the effects of integer and non-integer stress gradients in the constitutive relation and also to consider small-scale effect in the vibration of a Timoshenko nano-beam. The governing equation is solved by the Galerkin method.

Free vibration and buckling of a Timoshenko simply supported (S) nano-beam is investigated, and the influence of the fractional and nonlocal parameters is shown on the frequency ratio and buckling ratio. In this sense, the obtained formulation allows for an easier mapping of experimental results on nano-beams.

The new theory (fractional parameter) makes the modeling more flexible. The model can conclude all of the integer and non-integer operators and is not limited to the special operators such as ENT. In other words, it allows to use more sophisticated/flexible mathematics to model physical phenomena.

Key words: fractional calculus; nonlocal fractional derivative model; free vibration; Timoshenko beam; Galerkin method; buckling.

1. Introduction

Presence of small-scale effects in micro and nano-applications of the beam, plate, and shell-type structures makes size-dependent continuum theories to be widely used to model them due to their computational efficiency and simplicity in compression of an atomistic mechanic. There are different size-dependent continuum theories such as couple stress theory [1, 2], modified couple stress theory [3], strain gradient elasticity theory [4–7], nonlocal elasticity [8] and theories of material surfaces [9]. In recent years, nonlocal elasticity theory [7, 8] has been frequently used in different problems. The basic difference between classical elasticity and nonlocal elasticity is the definition of stress: in the local elasticity, the stress at a point depends only on the strain at this same point, whereas in the nonlocal elasticity stress at a point is a function of strains at all points in the continuum. This difference of stress definition leads to constitutive relation with an integral form which Eringen approximates with a differential form to make it easier to solve [8]. The Eringen approximation which only consists of the integer order of stress (2, 4, ...) agrees well with the dispersion curve of lattice dynamic. More recently [10], the generalized Eringen theory using fractional calculus showed that the fractional order of stress has a better agreement than the integer order.

Recently, Rahimi *et al.* [11] presented a general form of the Eringen nonlocal theory using fractional calculus which makes the modeling instrument more flexible and allows using to a greater degree the potential of mathematics in modeling of physical phenomena. Fractional calculus appeared first in 1695, and its application started to grow decades later in different fields such as mechanics, physics, electronics, wave propagation, control and viscoelastic studies [12–21]. It provides a new method for applications in mechanics problems and leads to fractional derivative models (FDMs). There are many works on introducing the FDMs and their application, however the idea to include a fractional term in the governing equation of the elastic problem has been proposed in [22]. In [23–25] a fractional nonlocal constitutive relation, introduced fractional strain gradient and presented fractional nonlocal Kirchhoff-love plate theory and fractional Euler-Bernoulli beam theory. It was shown that the new fractional model provides a good approximation of experimental data of Young modulus [25]. Atanackovic and Stankovic [26] by using Caputo fractional derivatives generalized wave equation in nonlocal elasticity. Demir *et al.* [27, 28] presented linear vibrations of axially moving systems which are modeled by a fractional derivative. They also studied the dynamic behavior of viscoelastic beam obeying a fractional differentiation constitutive law. Carpinteri *et al.* [29] by using the Caputo fractional derivative and changing the form of the attenuation function of strain presented a nonlocal elasticity model.

As mentioned, Challamel *et al.* [10] presented a general form of Eringen's theory by using the Caputo definition, which is in a better agreement with a dispersion curve of lattice dynamic versus the Eringen theory. One of the main difficulties encountered in the equation with fractional derivatives is a solution of such equations due to integral form of fractional derivatives definition (like Caputo, Riemann-Liouville and Grunwald-Letnikov), which limits application of the FDMs in different complex problems. So in the first part of the present work, we generalized the Eringen theory using conformal CFDD [30–32], which is the most natural and most helpful and has no integral form. Next, the presented FDM has been used to investigate the governing nonlinear equation of the motion of Timoshenko beam theory.

Timoshenko beam theory is a refined form of Euler-Bernoulli theory, which considers the effect of shear deformation, in addition to the effect of rotary inertia. Nonlocal Timoshenko beam is studied in different works. Wang *et al.* [33, 34] studied vibration and bending of nonlocal Timoshenko beam. They also studied buckling of micro and nano-rods and tubes based on the nonlocal Timoshenko beam theory. Wang *et al.* [35] and Ghannadpour and Mohammadi [36] also analyzed buckling of micro and nano-rods and tubes based on nonlocal Timoshenko beam theory but they used Chebyshev polynomials. Bending, buckling and free vibration of Timoshenko nano-beams have been studied in [35–39].

All of the works on Timoshenko beam which exist in the literature are based on the integer continuum model and when compared to our previous paper [11] the novelty of this work lies in the fact that the theory is based on the generalized form of the Eringen theory in which the generalization is based on the fractional calculus and CFD definition. In this theory, when the fractional parameter is equal to 2 the theory reduces to the classical form of nonlocal elasticity (Eringen's nonlocal theory). By contrast, the already published papers are based on the general form of strain energy in which when the fractional parameter is equal to 1 it reduces to the classical form of strain energy (classical local theory). So for the first time, free vibration and buckling of simply-supported Timoshenko nano-beam based on a fractional model have been investigated. The nonlinear governing equations were solved easily by the Galerkin method just like equations with integer derivatives, as this simple form of solution makes the FDM applicable to different complex problems.

2. Formulations

2.1. Conformable fractional derivatives definition

This definition, first presented in [30, 31] and then in [32], does not have integral form like other usual definitions such as Caputo, Riemann-Liouville and

Grunwald-Letnikov. It also eliminates some shortcomings of these solutions, and it is the most natural and simplest one. CFFD is as below (see Appendix):

$$
\begin{aligned}
D_x^\alpha(f)(x,y) &= x^{(\lceil\alpha\rceil-\alpha)}\frac{\mathrm{d}^{\lceil\alpha\rceil}f(x,y)}{\mathrm{d}x^{\lceil\alpha\rceil}}, \qquad & D_x^\alpha &= \frac{\partial^\alpha}{\partial x^\alpha},\\
D_y^\alpha(f)(x,y) &= y^{(\lceil\alpha\rceil-\alpha)}\frac{\mathrm{d}^{\lceil\alpha\rceil}f(x,y)}{\mathrm{d}y^{\lceil\alpha\rceil}}, \qquad & D_y^\alpha &= \frac{\partial^\alpha}{\partial y^\alpha},
\end{aligned}
\tag{2.1}
$$

where $n-1<\alpha\le n$ and $\lceil\alpha\rceil$ is the smallest integer equal or bigger than α. For example for $1<\alpha\le 2$ we have:

$$
\begin{aligned}
D_x^\alpha(f)(x,y) &= x^{(2-\alpha)}\frac{\mathrm{d}^2 f(x,y)}{\mathrm{d}x^2},\\
D_y^\alpha(f)(x,y) &= y^{(2-\alpha)}\frac{\mathrm{d}^2 f(x,y)}{\mathrm{d}y^2},
\end{aligned}
\tag{2.2}
$$

where for $\alpha=2$ it leads to (note that α cannot be equal to 1 based on $n-1<\alpha\le n$):

$$
\begin{aligned}
D_x^2(f)(x,y) &= \frac{\mathrm{d}^2 f(x,y)}{\mathrm{d}x^2},\\
D_y^2(f)(x,y) &= \frac{\mathrm{d}^2 f(x,y)}{\mathrm{d}y^2}.
\end{aligned}
\tag{2.3}
$$

2.2. The general form of Eringen's theory

The general form of Eringen's theory was first presented by CHALLAMEL *et al.* [10]. They used the Caputo definition in which the governed equation based on that had complex form due to the integral form of Caputo definition and their solution is difficult, so RAHIMI *et al.* [11] generalized it based on CFDD as below:

$$
\sigma-\mu^\alpha\frac{\mathrm{d}^\alpha\sigma}{\mathrm{d}x^\alpha}=E\varepsilon,
\tag{2.4}
$$

where α is a fractional parameter and it controls the order of stress in constitutive relation, so this model has one extra parameter in comparison to Eringen's theory which makes it more flexible. Here the fractional parameter is considered between 1 and 2 ($1<\alpha\le 2$). For this interval of α, the constitutive relation is as below:

$$
\sigma-\mu^{2-\alpha}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x^2}=E\varepsilon,
\tag{2.5}
$$

where for $\alpha=2$ it reduces to Eringen's model. Note that based on CFDD and the interval of α here α cannot be equal to 1.

2.3. Fractional nonlocal Timoshenko nano-beam

In this section, the equations of the motion of Timoshenko nano-beam were obtained based on the fractional nonlocal theory. The Euler-Lagrange equations are

$$\frac{\partial Q}{\partial x} + f - N\frac{\partial^2 w}{\partial x^2} = \rho \mathrm{A}\frac{\partial^2 w}{\partial x^2}, \tag{2.6}$$

$$\frac{\partial M}{\partial x} - Q = \rho I\frac{\partial^2 \varphi}{\partial t^2}. \tag{2.7}$$

First, eliminating Q between Eqs (2.6) and (2.7), leads to

$$\frac{\partial^2 M}{\partial x^2} = N\frac{\partial^2 w}{\partial x^2} - f + \rho \mathrm{A}\frac{\partial^2 w}{\partial t^2} + \rho I\frac{\partial^3 \varphi}{\partial t^3}. \tag{2.8}$$

And then the second derivative of Q is obtained from Eq. (2.6)

$$\frac{\partial^2 Q}{\partial x^2} = -\frac{\partial f}{\partial x} + N\frac{\partial^3 w}{\partial x^3} + \rho \mathrm{A}\frac{\partial^3 w}{\partial^2 t \partial x}. \tag{2.9}$$

In the Timoshenko beam theory, M and Q are obtained as below:

$$\begin{aligned} M = & -\mu^\alpha x^{2-\alpha} f + \mu^\alpha x^{2-\alpha} N\frac{\partial^2 w}{\partial x^2} + \rho A\mu^\alpha x^{2-\alpha}\frac{\partial^2 w}{\partial t^2} \\ & + \rho I\mu^\alpha x^{2-\alpha}\frac{\partial^3 \varphi}{\partial t^2 \partial x} + EI\frac{\partial \varphi}{\partial x}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} Q = & -\mu^\alpha x^{2-\alpha}\frac{\partial f}{\partial x} + \mu^\alpha x^{2-\alpha} N\frac{\partial^3 w}{\partial x^3} \\ & + \rho A\mu^\alpha x^{2-\alpha}\frac{\partial^3 w}{\partial t^2 \partial x} + KAG\left(\varphi + \frac{\partial w}{\partial x}\right). \end{aligned} \tag{2.11}$$

Substituting Eqs (2.10) and (2.11) into Eqs (2.6) and (2.7) leads to the equation of the motion of Timoshenko beam under uniform transverse and axial forces:

$$\begin{aligned} & -\mu^\alpha x^{2-\alpha}\frac{\partial^2 f}{\partial x^2} - \mu^\alpha N(2-\alpha)x^{1-\alpha}\frac{\partial f}{\partial x} + \mu^\alpha N x^{2-\alpha}\frac{\partial^4 w}{\partial x^4} \\ & + \mu^\alpha N(2-\alpha)x^{1-\alpha}\frac{\partial^3 w}{\partial x^3} + \rho A\mu^\alpha x^{2-\alpha}\frac{\partial^4 w}{\partial t^2 \partial x^2} + \rho A\mu^\alpha x^{1-\alpha}(2-\alpha)\frac{\partial^3 w}{\partial t^2 \partial x} \\ & + KAG\left(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + f - N\frac{\partial^2 w}{\partial x^2} = \rho A\frac{\partial^2 w}{\partial t^2}, \end{aligned} \tag{2.12}$$

$$-\mu^{\alpha}x^{1-\alpha}(2-\alpha)f+\mu^{\alpha}x^{1-\alpha}(2-\alpha)N\frac{\partial^2 w}{\partial x^2}+\rho A\mu^{\alpha}x^{1-\alpha}(2-\alpha)\frac{\partial^2 w}{\partial t^2} \\ +\rho I\mu^{\alpha}x^{2-\alpha}\frac{\partial^4\varphi}{\partial t^2\partial x^2}+\rho I\mu^{\alpha}(2-\alpha)x^{1-\alpha}\frac{\partial^3\varphi}{\partial t^2\partial x}+EI\frac{\partial^2\varphi}{\partial x^2} \\ -KAG\left(\varphi+\frac{\partial w}{\partial x}\right)=\rho I\frac{\partial^2\varphi}{\partial t^2}. \tag{2.13}$$

Note that where $\alpha = 2$ it reduces to the equation based on Eringen's nonlocal theory and where $\mu = 0$ it becomes equation based on the classic theory.

2.3.1. Free vibration. The non-dimensional non-linear free vibration equation has been obtained as follows by setting transverse and axial force to zero

$$\mu^{\alpha}x^{2-\alpha}\frac{\partial^4 w}{\partial x^4}+\rho A\mu^{\alpha}x^{2-\alpha}\frac{\partial^4 w}{\partial t^2\partial x^2}+\rho A\mu^{\alpha}x^{1-\alpha}(2-\alpha)\frac{\partial^3 w}{\partial t^2\partial x} \\ +KAG\left(\frac{\partial\varphi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)=\rho A\frac{\partial^2 w}{\partial t^2}, \tag{2.14}$$

$$\rho A\mu^{\alpha}x^{1-\alpha}(2-\alpha)\frac{\partial^2 w}{\partial t^2}+\rho I\mu^{\alpha}x^{2-\alpha}\frac{\partial^4\varphi}{\partial t^2\partial x^2} \\ +\rho I\mu^{\alpha}(2-\alpha)x^{1-\alpha}\frac{\partial^3\varphi}{\partial t^2\partial x}+EI\frac{\partial^2\varphi}{\partial x^2}-KAG\left(\varphi+\frac{\partial w}{\partial x}\right)=\rho I\frac{\partial^2\varphi}{\partial t^2}. \tag{2.15}$$

2.3.2. Buckling. The non-dimensional buckling equation has been obtained as follows by setting inertia terms and transverse force to zero

$$\mu^{\alpha}Nx^{2-\alpha}\frac{\partial^4 w}{\partial x^4}+\mu^{\alpha}N(2-\alpha)x^{1-\alpha}\frac{\partial^3 w}{\partial x^3}+KAG\left(\frac{\partial\varphi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)-N\frac{\partial^2 w}{\partial x^2}=0, \tag{2.16}$$

$$\mu^{\alpha}x^{1-\alpha}(2-\alpha)N\frac{\partial^2 w}{\partial x^2}+EI\frac{\partial^2\varphi}{\partial x^2}-KAG\left(\varphi+\frac{\partial w}{\partial x}\right)=0. \tag{2.17}$$

3. Numerical solutions

One of the main problems of differential equations based on the fractional calculus is the numerical solution of these equations due to the integral forms of former fractional definition such as Caputo, Riemann-Liouville and Grunwald-Letnikov. Therefore, some researches by developing numerical methods or introducing new methods tried to present a numerical solution [40–46]. Using the CFDD enables to solve the non-linear governed equations easily by the Galerkin method just like equations with derivatives of integer order. This is a positive point of the FDM which has been presented here. This FDM can be used in different complex problems due to the simple form of a numerical solution.

3.1. *Free vibration*

For free vibration, the transverse displacement is considered as below:

$$(3.1)\quad \begin{aligned} w(x,t) &= \sum_{i=1}^{\infty} q_1(t)\chi_i(x) \approx \sum_{i=1}^{M} q_1(t)\chi_i(x) \approx \sum_{i=1}^{M} q_1(t)\sin(n\pi x/L),\\ \varphi(x,t) &= \sum_{i=1}^{\infty} q_2(t)\kappa_N(x) \approx \sum_{i=1}^{K} q_2(t)\kappa_N(x) \approx \sum_{i=1}^{K} q_2(t)\cos(n\pi x/L),\end{aligned}$$

where $x = x/L$, $t = t/t_0$, $t_0 = \sqrt{\rho AL^4/EI}$, $q(t)$ is time function to be calculated, χ_i and κ_i are suitable shape function and considered as a classic form of nonlocal theory (Eringen's theory) [37]. Substituting Eq. (3.1) into Eqs (2.14) and (2.15) leads to:

$$(3.2)\quad \begin{aligned} &\mu^\alpha L^{-3}(Lx)^{2-\alpha}\sum_{i=1}^{N} q_1(t)\chi_i^{(4)}(x) + \rho A\mu^\alpha t_0^{-2}L^{-1}(Lx)^{2-\alpha}\sum_{i=1}^{N} q_1^{(2)}(t)\chi_i^{(2)}(x)\\ &\qquad + \rho A\mu^\alpha t_0^{-2}(Lx)^{1-\alpha}(2-\alpha)\sum_{i=1}^{N} q_1^{(2)}(t)\chi_i^{(1)}(x)\\ &+ KAGL^{-1}\left(\sum_{i=1}^{N} q_2(t)\kappa_i^{(1)}(x) + \sum_{i=1}^{N} q_1(t)\chi_i^{(2)}(x)\right) = \rho ALt_0^{-2}\sum_{i=1}^{N} q_1^{(2)}(t)\chi_i(x),\end{aligned}$$

$$(3.3)\quad \begin{aligned} &\rho A\mu^\alpha Lt_0^{-2}(Lx)^{1-\alpha}L(2-\alpha)\sum_{i=1}^{N} q_1^{(2)}(t)\chi_i(x)\\ &\qquad + \rho I\mu^\alpha L^{-2}t_0^{-2}(Lx)^{2-\alpha}\sum_{i=1}^{N} q_2^{(2)}(t)\kappa_i^{(2)}(x)\\ &+ \rho I\mu^\alpha(2-\alpha)L^{-1}t_0^{-2}(Lx)^{1-\alpha}\sum_{i=1}^{N} q_2^{(2)}(t)\kappa_i^{(1)}(x) + EIL^{-2}\sum_{i=1}^{N} q_2(t)\kappa_i^{(2)}(x)\\ &- KAG\left(\sum_{i=1}^{N} q_2(t)\kappa_i(x) + \sum_{i=1}^{N} q_1(t)\chi_i^{(1)}(x)\right) = \rho It_0^2\sum_{i=1}^{N} q_2^{(2)}(t)\kappa_i(x).\end{aligned}$$

Multiplying Eq. (3.2) into the $\chi_j(x)$ and Eq. (3.3) into the $\kappa_j(x)$, integrating outcome from 0 to 1 and considering $q_1(t) = ae^{iwt}$ and $q_2(t) = be^{iwt}$ lead to:

$$[(A_1 + A_2) + (A_3 + A_4 + A_5)\omega^2]\, a + [A_6]b = 0, \tag{3.4}$$

$$[B_1 + B_2\omega^2]a + [B_3 + B_4 + (B_5 + B_6 + B_7)\omega^2]b = 0, \tag{3.5}$$

where

$$A_1 = \int_0^1 \mu^\alpha L^{-3}(Lx)^{2-\alpha}\chi_i^{(4)}(x)\chi_i(x)\,\mathrm{d}x,$$

$$A_2 = \int_0^1 KAGL^{-1}\chi_i^{(2)}(x)\chi_j(x)\,\mathrm{d}x,$$

$$A_3 = \int_0^1 \rho A\mu^\alpha t_0^{-2}(Lx)^{1-\alpha}(2-\alpha)\chi_i^{(1)}(x)\chi_i(x)\,\mathrm{d}x,$$

$$A_4 = \int_0^1 \rho A\mu^\alpha t_0^{-2}L^{-1}(Lx)^{2-\alpha}\chi_i^{(2)}(x)\chi_j(x)\,\mathrm{d}x,$$

$$A_5 = \int_0^1 -\rho ALt_0^{-2}\chi_i(x)\chi_j(x)\,\mathrm{d}x,$$

$$A_6 = \int_0^1 KAGL^{-1}\kappa_i^{(1)}(x)\chi_j(x)\,\mathrm{d}x,$$

$$B_1 = \int_0^1 -KAG\chi_i^{(1)}(x)\kappa_j(x)\,\mathrm{d}x,$$

$$B_2 = \int_0^1 \rho A\mu^\alpha Lt_0^{-2}(Lx)^{1-\alpha}L(2-\alpha)\chi_i(x)\kappa_j(x)\,\mathrm{d}x,$$

$$B_3 = \int_0^1 -KAG\kappa_i(x)\kappa_j(x)\,\mathrm{d}x,$$

$$B_4 = \int_0^1 EIL^{-2}\kappa_i^{(2)}(x)\kappa_j(x)\,\mathrm{d}x,$$

$$B_5 = \int_0^1 \rho I\mu^\alpha L^{-2}t_0^{-2}(Lx)^{2-\alpha}\kappa_i^{(2)}(x)\kappa_j(x)\,\mathrm{d}x,$$

$$B_6 = \int_0^1 \rho I\mu^\alpha (2-\alpha)L^{-1}t_0^{-2}(Lx)^{1-\alpha}\kappa_i^{(1)}(x)\kappa_j(x)\,\mathrm{d}x,$$

$$B_7 = \int_0^1 -\rho It_0^2\kappa_i(x)\kappa_j(x)\,\mathrm{d}x.$$

Equations (3.4) and (3.5) can be rewritten in the matrix form as below:

$$\begin{bmatrix} (A_1+A_2)+(A_3+A_4+A_5)\omega^2 & A_6 \\ B_1+B_2\omega^2 & B_3+B_4+(B_5+B_6+B_7)\omega^2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

3.2. Buckling

For buckling, the transverse displacement is considered as below:

$$w(x,t) \approx \sum_{i=1}^{N} \frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i(x) = \sum_{i=1}^{N} \frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\sin\Big(\frac{i\pi x}{L}\Big), \tag{3.6}$$

$$\varphi(x,t) \approx \sum_{i=1}^{N} \frac{\lambda_i L^3}{n^3\pi^3}\kappa_i(x) = \sum_{i=1}^{N} \frac{\lambda_i L^3}{n^3\pi^3}\cos\Big(\frac{i\pi x}{L}\Big), \tag{3.7}$$

where

$$\Lambda = (1+n^2\pi^2\Omega), \qquad \Omega = \frac{EI}{GAKL^2}.$$

Substituting Eqs (3.6) and (3.7) into Eqs (2.16) and (2.17) leads to:

$$\begin{aligned} \mu^\alpha x^{2-\alpha}N\sum_{i=1}^{N}\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(4)}(x) + \mu^\alpha N(2-\alpha)x^{1-\alpha}\sum_{i=1}^{N}\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(3)}(x) \\ + KAG\Big(\sum_{i=1}^{N}\frac{\lambda_i L^3}{n^3\pi^3}\kappa_i^{(1)}(x) + \sum_{i=1}^{N}\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(2)}(x)\Big) \\ - N\sum_{i=1}^{N}\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(2)}(x) = 0, \end{aligned} \tag{3.8}$$

$$\mu^{\alpha}x^{1-\alpha}(2-\alpha)N\sum_{i=1}^{N}\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(2)}(x)+EI\sum_{i=1}^{N}\frac{\lambda_i L^3}{n^3\pi^3}\kappa_i^{(2)}(x) - KAG\left(\sum_{i=1}^{N}\frac{\lambda_i L^3}{n^3\pi^3}\kappa_i(x)+\sum_{i=1}^{N}\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(1)}(x)\right)=0. \tag{3.9}$$

Multiplying Eq. (3.8) into the$\chi_j(x)$and Eq. (3.9) into the $\kappa_j(x)$ and integrating outcome from 0 to L lead to:

$$\begin{aligned}\mu^{\alpha}x^{2-\alpha}N\int_0^L\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(4)}(x)\chi_j(x)\,\mathrm{d}x &+\mu^{\alpha}N(2-\alpha)x^{1-\alpha}\int_0^L\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(3)}(x)\chi_j(x)\,\mathrm{d}x\\ &+KAG\left(\int_0^L\frac{\lambda_i L^3}{n^3\pi^3}\kappa_i^{(1)}(x)\chi_j(x)\,\mathrm{d}x+\int_0^L\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(2)}(x)\chi_j(x)\,\mathrm{d}x\right)\\ &-N\int_0^L\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(2)}(x)\chi_j(x)\,\mathrm{d}x=0,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\mu^{\alpha}x^{1-\alpha}(2-\alpha)N\int_0^L\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(2)}(x)\kappa_j(x)\,\mathrm{d}x&+EI\int_0^L\frac{\lambda_i L^3}{n^3\pi^3}\kappa_i^{(2)}(x)\kappa_j(x)\,\mathrm{d}x\\ -KAG\left(\int_0^L\frac{\lambda_i L^3}{n^3\pi^3}\kappa_i(x)\kappa_j(x)\,\mathrm{d}x\right.&\left.+\int_0^L\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(1)}(x)\kappa_j(x)\,\mathrm{d}x\right)=0,\end{aligned} \tag{3.11}$$

$$\begin{aligned}&KAG\left(\int_0^L\frac{\lambda_i L^3}{n^3\pi^3}\kappa_i^{(1)}(x)\chi_j(x)\,\mathrm{d}x+\int_0^L\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(2)}(x)\chi_j(x)\,\mathrm{d}x\right)\\ &=KAG\left(\frac{n\pi}{L}\right)\left(\int_0^L\frac{\lambda_i L^3}{n^3\pi^3}\kappa_i(x)\chi_j(x)\,\mathrm{d}x+\int_0^L\frac{\lambda_i\Lambda_i L^4}{n^4\pi^4}\chi_i^{(1)}(x)\chi_j(x)\,\mathrm{d}x\right).\end{aligned} \tag{3.12}$$

According to Eqs (3.11) and (3.12) the following is obtained in Eq. (3.13):

$$
\begin{aligned}
\mu^\alpha x^{2-\alpha} N \int_0^L \frac{\lambda_i \Lambda_i L^4}{n^4\pi^4}\chi_i^{(4)}(x)\chi_j(x)\,\mathrm{d}x
&+ \mu^\alpha N(2-\alpha)x^{1-\alpha}\int_0^L \frac{\lambda_i \Lambda_i L^4}{n^4\pi^4}\chi_i^{(3)}(x)\chi_j(x)\,\mathrm{d}x \\
&+ \left(\frac{n\pi}{L}\right)\left(\mu^\alpha x^{1-\alpha}(2-\alpha)N\int_0^L \frac{\lambda_i \Lambda_i L^4}{n^4\pi^4}\chi_i^{(2)}(x)\kappa_j(x)\,\mathrm{d}x\right. \\
&\left.+EI\int_0^L \frac{\lambda_i L^3}{n^3\pi^3}\kappa_i^{(2)}(x)\kappa_j(x)\,\mathrm{d}x\right) - N\int_0^L \frac{\lambda_i \Lambda_i L^4}{n^4\pi^4}\chi_i^{(2)}(x)\chi_j(x)\,\mathrm{d}x = 0.
\end{aligned}
\tag{3.13}
$$

The critical buckling load is given by

$$
N = \frac{\hbar_1}{\lambda\!\!\!^{-}_1 + \lambda\!\!\!^{-}_2 + \lambda\!\!\!^{-}_3 + \lambda\!\!\!^{-}_4},
$$

where

$$
\begin{aligned}
\lambda\!\!\!^{-}_1 &= \int_0^L \mu^\alpha x^{2-\alpha}\frac{\lambda_i \Lambda_i L^4}{n^4\pi^4}\chi_i^{(4)}(x)\chi_j(x)\,\mathrm{d}x,\\
\lambda\!\!\!^{-}_2 &= \int_0^L \mu^\alpha (2-\alpha)x^{1-\alpha}\frac{\lambda_i \Lambda_i L^4}{n^4\pi^4}\chi_i^{(3)}(x)\chi_j(x)\,\mathrm{d}x,\\
\lambda\!\!\!^{-}_3 &= \int_0^L \left(\frac{n\pi}{L}\right)\mu^\alpha x^{1-\alpha}(2-\alpha)\frac{\lambda_i \Lambda_i L^4}{n^4\pi^4}\chi_i^{(2)}(x)\kappa_j(x)\,\mathrm{d}x,\\
\lambda\!\!\!^{-}_4 &= \int_0^L -\frac{\lambda_i \Lambda_i L^4}{n^4\pi^4}\chi_i^{(2)}(x)\chi_j(x)\,\mathrm{d}x,\\
\hbar_1 &= \int_0^L \left(\frac{n\pi EI}{L}\right)\frac{\lambda_i L^3}{n^3\pi^3}\kappa_i^{(2)}(x)\kappa_j(x)\,\mathrm{d}x.
\end{aligned}
$$

4. Results

Free vibration and buckling of the nano-Timoshenko beam have been studied based on the general form of Eringen's nonlocal theory. In this section, the results are presented and validated by making two comparisons with the results from other studies. In Table 1, the first three natural frequencies are compared with the results of Rao [47], where the nano-beam has the following materials properties: $L = 1$, $h/L = 0.15$, $E = 207 \cdot 10^9$, $\rho = 76.5 \cdot 10^3$, $K = 5/6$. Then in Table 2, the dimensionless first natural frequencies for different values of the nonlocal parameter are compared with the results of Reddy [37]. The dimensionless first natural frequency and the dimensionless buckling load for different nonlocal parameters are compared with the results of Reddy in Tables 2 and 3 respectively. As it can be observed our results agree well with the results of Reddy [37] and Rao [47].

Table 1. Comparison of the first three natural frequencies.

Natural frequency [rad/s]	Ref. [47]		Present	
	Bending	Shear	Bending	Shear
$N = 1$	677.8909	22259.102	677.8829	$2.2259 \cdot 10^4$
$N = 2$	2473.3691	24402.975	$2.4733 \cdot 10^3$	$2.4403 \cdot 10^4$
$N = 3$	4948.0036	27446.279	$4.9479 \cdot 10^3$	$2.7447 \cdot 10^4$

Table 2. Comparison of the non-dimensional first natural frequency: $L = 10$, $\rho = 1$, $L/h = 100$, $E = 30 \cdot 10^6$.

Nonlocal parameter (μ^2)	Ref. [37]	Present
0.0	9.8671	9.8679
0.5	9.4031	9.7435
1.0	8.9807	9.3954
1.5	8.5947	8.8662
2.0	8.2405	8.2884

Table 3. Comparison of the non-dimensional buckling ratio: $L = 10$, $\rho = 1$, $L/h = 100$, $E = 30 \cdot 10^6$.

Nonlocal parameter (μ^2)	Ref. [37]	Present
0.0	9.8671	9.8679
0.5	9.4031	9.7435
1.0	8.9807	9.3954
1.5	8.5947	8.8662
2.0	8.2405	8.2884

Figure 1 shows the effects of aspect ratio on the natural frequencies, for different values of the fractional parameter. Results are given for $e_0 a = 0.5$. It is seen for all values of the fractional parameter ($\alpha = 2$, 1.8, 1.6, 1.4, 1.2) that

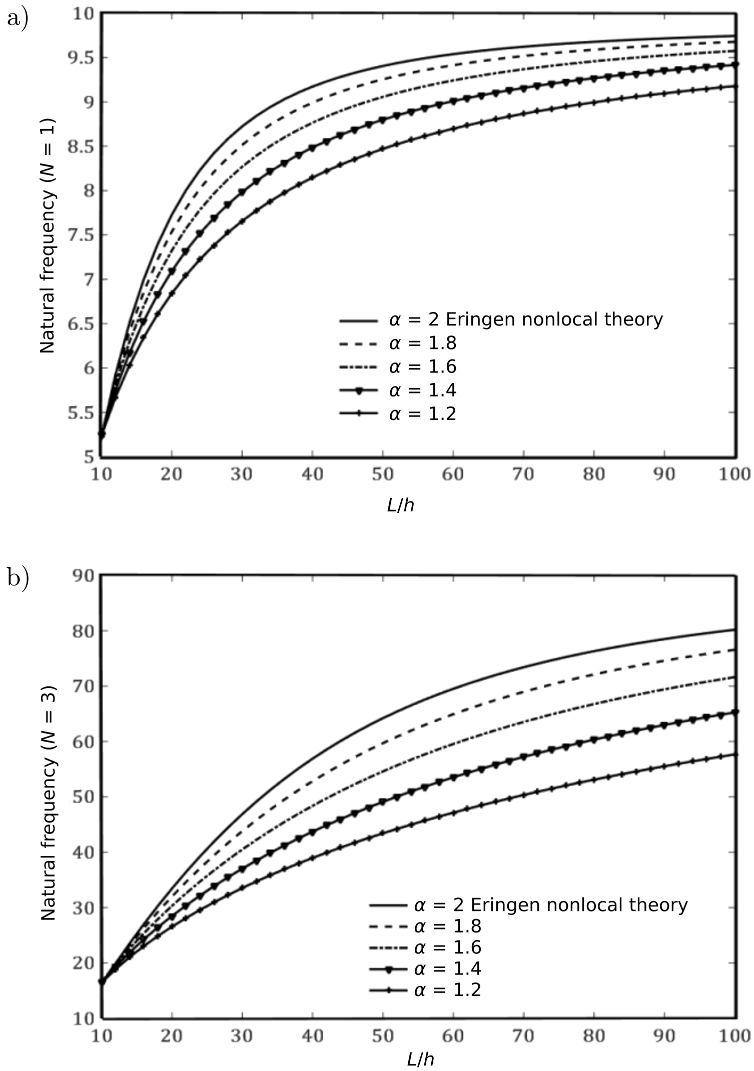

FIG. 1. Natural frequency versus aspect ratio where fractional parameter is $\alpha = 2$, 1.8, 1.6, 1.4, 1.2 and the nonlocal parameter is $e_0a = 0.5$: a) $N = 1$, b) $N = 3$.

by decreasing the aspect ratio the values of the natural frequency are decreased. This result for $\alpha = 2$ (Eringen's nonlocal theory) agrees very well with the result in REDDY [37]. In Fig. 2, the dimensionless frequency is shown versus both the fractional parameter and the aspect ratio. In Fig. 2, it is observed that by increasing the aspect ratio, the effects of the fractional parameter on the natural frequency are increased. This is also more noticeable for the second and third natural frequency in Fig. 1.

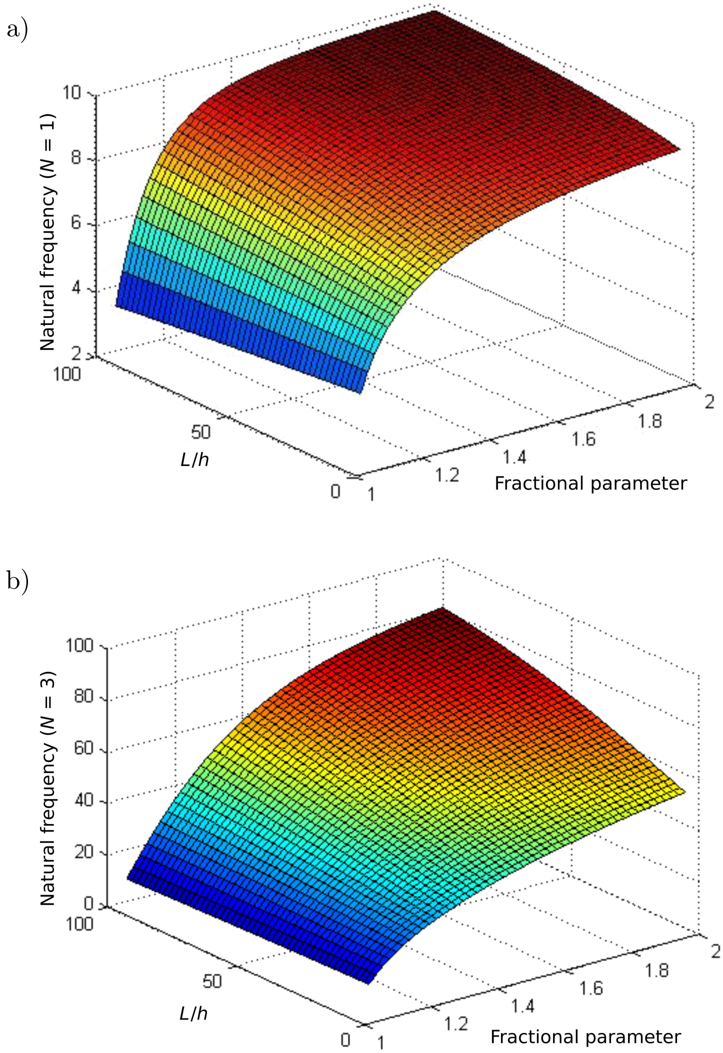

FIG. 2. Natural frequency versus aspect ratio and fractional parameter where the nonlocal parameter is $e_0a = 0.5$: a) $N = 1$, b) $N = 3$.

In Table 4 results are given for various values of the aspect ratio ($L/h = 100$, 50), the nonlocal parameter ($e_0a = 1$, 2, 3) and the fractional parameter ($\alpha = 2$, 1.8, 1.6, 1.4, 1.2).

Figure 3 shows the variation of buckling response of the nano-beam with the aspect ratio. The nano-beam has the following material properties: $E = 30 \cdot 10^6$, $L = 10$, $\rho = 1$. The local results are given for $e_0a = 0$. In this example, the aspect ratio varies from 10 to 100. It is seen from this figure that the nonlocal solution

Table 4. First three non-dimensional natural frequencies for different values of fractional and nonlocal parameters and aspect ratio.

L/h	Nonlocal parameter (e_0a)	N	Fractional parameter (α)				
			2	1.8	1.6	1.4	1.2
100	1	1	9.3954	9.2461	9.0488	8.7920	8.4647
	2		8.2884	8.0868	7.8545	7.5909	7.2963
	3		7.0507	6.9256	6.7788	6.6123	6.4280
	1	2	33.3381	31.8400	30.0148	27.8678	25.4446
	2		24.3681	23.2362	21.9718	20.6006	19.1537
	3		18.1528	17.7029	17.1316	16.4603	15.7111
	1	3	64.4136	59.8924	54.7861	49.2700	43.5751
	2		41.2306	38.9093	36.3489	33.6385	30.8634
	3		29.0346	28.2741	27.2611	26.0564	24.7169
50	1	1	8.3345	8.1234	7.8828	7.6122	7.3117
	2		6.0920	6.0386	5.9711	5.8909	5.7989
	3		4.5382	4.6425	4.7322	4.8085	4.8727
	1	2	24.4667	23.3064	22.0181	20.6276	19.1653
	2		14.4383	14.3569	14.1733	13.9012	13.5546
	3		9.9105	10.2558	10.5240	10.7199	10.8495
	1	3	41.2938	38.9300	36.3352	33.5987	30.8050
	2		22.4535	22.3716	22.0791	21.6125	21.0082
	3		15.0928	15.6788	16.1176	16.4218	16.6070

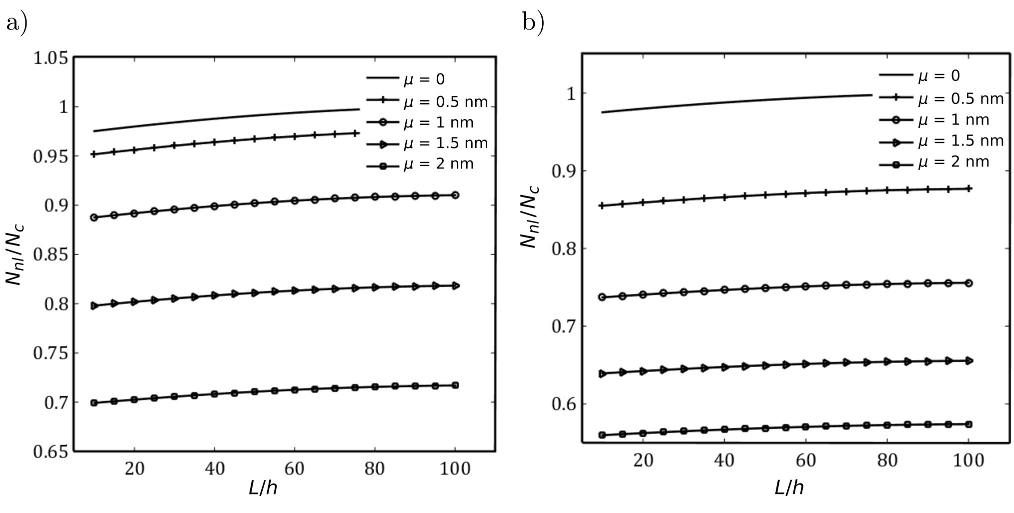

FIG. 3. The effect of aspect ratio on the buckling ratio for: a) $\alpha = 2$, b) $\alpha = 1.2$.

of the buckling load is smaller than the local buckling load due to the small scale effects. Additionally, it is observed that by decreasing the fractional parameter, the buckling load is decreased. This result for $\alpha = 2$ (Eringen's nonlocal theory) agrees very well with the result of REDDY [37]. Moreover, it can be observed in each diagram that the buckling load due to increasing the nonlocal parameter is less reduced, and that by increasing the aspect ratio, the buckling load is increased. An interesting result observed in Fig. 3 is that the effect of nonlocal parameter on buckling load is negligible for the aspect ratios less than 20.

Figure 4 shows the effect of nonlocal parameter on the buckling load with the aspect ratio of 10, 50 and 100 respectively for various values of the fractional parameter, and nonlocal parameter varying from 0 to 2. One can observe that by increasing the nonlocal parameter, dimensionless buckling load is decreased. It can be concluded that modeling based on the fractional parameter is more suitable than the nonlocal parameter, as the fractional parameter offers many approximations for the nano-sized structures.

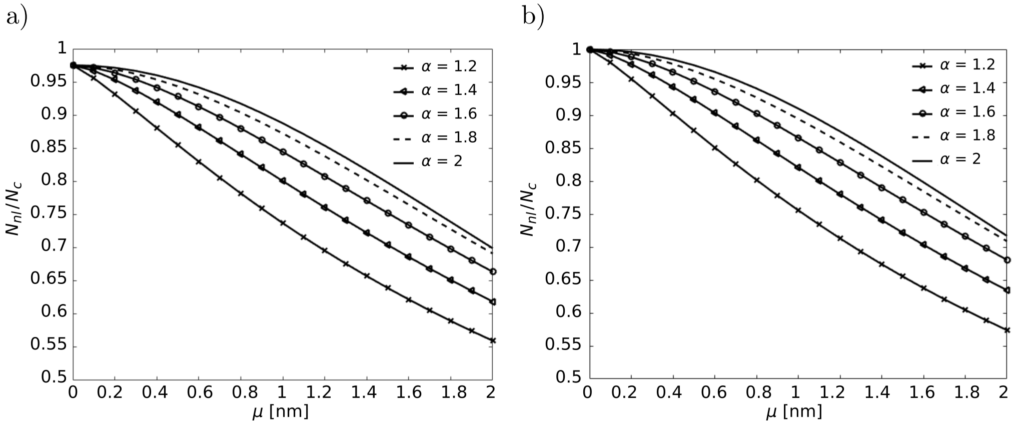

FIG. 4. The effect of nonlocal parameter on the buckling ratio for: a) $L/h = 10$, b) $L/h = 100$.

5. CONCLUSIONS

A fractional nonlocal model is used that it is the general form of the Eringen nonlocal theory and has two free parameters:

1) the nonlocal parameter which used to consider small scale effects in micron and sub-micron,
2) the fractional parameter used to control the order of stress gradient in constitutive relation which by controlling the contribution of stress makes the modeling instrument more fixable.

There are different definitions of the fractional derivatives, in which the level of difficulty of the solution method due to the integral form of them leads to using

CFDD here. The CFDD enables to solve the governing equation like equations of integer order due to the simple form and the absence of integral form in its definition. The simple form of the numerical solution is the main positive point of the fractional model. This property makes it be applicable to different complex problems. The fractional model was used to study free vibration of a Timoshenko nano-beam in which the coupled non-linear governed equations were solved easily using the Galerkin method.

The first three dimensionless natural frequencies ($N = 1$, $N = 2$, $N = 3$) and buckling load were studied based on the fractional model. For all of them ($N = 1$, $N = 2$, and $N = 3$), by decreasing the fractional parameter from its classic value ($\alpha = 2$ which is Eringen's nonlocal theory) the values of the frequencies and buckling load decreased, and due to increasing the nonlocal parameter the buckling mode was less reduced. In addition, by increasing the order of natural frequency, the effects of increasing the nonlocal parameter on the natural frequency is decreased. On the other hand, for all values of the fractional parameter by decreasing the aspect ratio the values of natural frequency and buckling load are decreased and by increasing the aspect ratio, the effects of the fractional parameter on the natural frequency and buckling load are increased. Numerical results show that the fractional effects play an important role in the buckling responses of the nano-beam. Furthermore, it is found that the fractional parameter has a great influence on the nano-beam, and the responses can be controlled by choosing proper values of the fractional parameter.

Appendix

CFDD for multi-variables functions:

Assume the function $f(x, y)$, we have:

$$\text{(A1)} \qquad \begin{aligned} f_x(x,y) &= \frac{\mathrm{d}f(x,y)}{\mathrm{d}x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}, \\ f_y(x,y) &= \frac{\mathrm{d}f(x,y)}{\mathrm{d}y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}. \end{aligned}$$

Based on CFDD we have [23]:

$$\text{(A2)} \qquad \begin{aligned} f_x^{\alpha}(x,y) &= \frac{\mathrm{d}^{\alpha}f(x,y)}{\mathrm{d}x^{\alpha}} = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon x^{1-\alpha},y)-f(x,y)}{\varepsilon}, \\ f_y^{\alpha}(x,y) &= \frac{\mathrm{d}^{\alpha}f(x,y)}{\mathrm{d}y^{\alpha}} = \lim_{\varepsilon\to 0}\frac{f(x,y+\varepsilon y^{1-\alpha})-f(x,y)}{\varepsilon}. \end{aligned}$$

If $0 < \alpha \leq 1$, let $h = \varepsilon x^{\alpha-1}$, $h = \varepsilon y^{\alpha-1}$ then Eq. (A2) is:

$$\begin{aligned}
f_x^\alpha(x,y) &= \frac{\mathrm{d}^\alpha f(x,y)}{\mathrm{d}x^\alpha} = \lim_{\varepsilon\to 0} \frac{f(x+\varepsilon x^{1-\alpha},y) - f(x,y)}{\varepsilon} \\
&= x^{1-\alpha} \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = x^{1-\alpha} \frac{\mathrm{d}f(x,y)}{\mathrm{d}x}, \\
f_y^\alpha(x,y) &= \frac{\mathrm{d}^\alpha f(x,y)}{\mathrm{d}y^\alpha} = \lim_{\varepsilon\to 0} \frac{f(x,y+\varepsilon y^{1-\alpha}) - f(x,y)}{\varepsilon} \\
&= y^{1-\alpha} \lim_{h\to 0} \frac{f(x,y) - f(x,y+h)}{h} = y^{1-\alpha} \frac{\mathrm{d}f(x,y)}{\mathrm{d}y}.
\end{aligned} \tag{A3}$$

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